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THE QUARTERLY JOURNAL OF MECHANICS AND APPLIED MATHEMATICS

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AXIALLY SYMMETRIC STAGNATION POINT FLOW WITH HEAT TRANSFER IN MAGNETOHYDRODYNAMICS

By G. POOTIS and L. SOWERBY

(Department of Theoretical Mechanics, University of Bristol)

[Received 9 July 1959. Revised received 7 April 1960]

SUMMARY

The steady axially symmetric stagnation point flow of an incompressible electrically conducting viscous fluid in the presence of a magnetic field normal to the wall is investigated. The wall is assumed to be thermally insulated and all physical properties of the fluid such as viscosity, electrical conductivity, and magnetic permeability, etc., are assumed to be independent of the temperature and the strength of the magnetic field. Solution of the equations of magnetohydrodynamics leads to the conclusion of the existence of three regions of flow: (i) a layer of fluid near the wall in which viscosity is important, called the magneto-viscous layer having thickness of order $(\nu/a)^{1/2}$, (ii) a buffer layer in which viscosity is unimportant, called the magneto-inviscid layer and having thickness of order $1/(8\pi\sigma\mu_e a)^{1/2}$, and (iii) a region of potential flow in which the velocity field is directly proportional in magnitude to the magnetic field. Here a is a parameter of dimensions (time)^{1/2}, such that the radial component of external velocity is ax .

A numerical solution of the basic equations has been obtained for the special case of $\beta = \lambda/2\nu = 10^6$ and $P = 0.72$, in the form of a series expansion in terms of the magnetic parameter $m = \mu_e H_0^2(2a)/(p\nu)$. In particular it appears that the presence of the magnetic field produces, in the vicinity of the stagnation point, a considerable reduction in the local shear stress and the entemperature at the wall. The new results obtained are intended to supplement the results of Neuringer and McHoy (1), Rossow (2), and Meyer (3) who have investigated the corresponding problem of the two-dimensional stagnation point flow.

Symbols

E.M.U. and C.G.S. system used throughout.

E electric intensity.

H $(H_x, 0, H_z)$ magnetic intensity.

j current density.

σ electrical conductivity.

μ_e magnetic permeability.

q $(u_r, 0, u_z)$ fluid velocity.

ρ fluid density.

μ dynamic viscosity.

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Symbols

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j current density.

σ electrical conductivity.

μ_e magnetic permeability.

$\mathbf{q} = (u_r, 0, u_z)$ fluid velocity.

ρ fluid density.

μ dynamic viscosity.

- $\nu = \mu/\rho$ kinematic viscosity.
 a parameter of dimensions (time)⁻¹.
 p pressure.
 T temperature.
 k thermometric conductivity.
 c_p specific heat at constant pressure.
 Q viscous heat dissipation.
 θ, Θ dimensionless functions describing the local temperature distributions in the magneto-viscous and magneto-inviscid layers respectively.
 P Prandtl number.
 $\lambda = 1/4\pi\sigma\mu_e$ magnetic viscosity.
 $\beta = 1/8\pi\sigma\mu_e\nu$ a parameter.
 $m = \mu_e^2 H_w^2(2\sigma)/\rho a$ magnetic parameter.
 f, g, h dimensionless functions of $\zeta = (a/\nu)^{1/2}z$ relevant to the magneto-viscous layer.
 F, G, H dimensionless functions of $\eta = (8\pi\sigma\mu_e a)^{1/2}(z+A)$ relevant to the magneto-inviscid layer.
 (r, ϕ, z) cylindrical polar coordinates.

Subscripts

- w body surface.
 ∞ free stream.

1. Introduction

RECENT papers by Neuringer and McIlroy (1), Rossow (2), and Meyer (3) deal with the reduction of shear stress and heat transfer rates due to the influence of a magnetic field on two-dimensional stagnation point flow. The present paper deals with the magnetohydrodynamic effects on axially symmetric stagnation point flow and heat transfer. For simplicity, as in references (1) to (3), we shall consider a viscous incompressible electrically conducting fluid, and all physical properties of the fluid such as viscosity, electrical conductivity, magnetic permeability, etc., are assumed to be independent of the temperature and the strength of the magnetic field. Cylindrical polar coordinates (r, ϕ, z) are used, the wall coincides with the plane $z = 0$, and the stagnation point is at the origin. Also

- (a) the longitudinal flow is taken to be in the negative z -direction,
 (b) the magnetic field at the wall is taken normal to the wall, and
 (c) the wall is thermally insulated, i.e. $(\partial T/\partial z)_{z=0} = 0$.

The basic equations for axially symmetric flow in E.M.U. and C.G.S. units are, in the usual notation:

$$\operatorname{div} \mathbf{q} = 0, \quad (1.1)$$

$$\frac{D\mathbf{q}}{Dt} + \frac{1}{\rho} \operatorname{grad} p = \nu \nabla^2 \mathbf{q} + \frac{\mu_e}{\rho} \mathbf{j} \times \mathbf{H}, \quad (1.2)$$

$$c_p \frac{DT}{Dt} - \frac{\mathbf{j}^2}{\rho \sigma} - Q = \frac{k}{\rho} \nabla^2 T, \quad (1.3)$$

together with the Maxwell electromagnetic field equations,

$$\operatorname{div} \mathbf{H} = 0, \quad (1.4)$$

$$\operatorname{curl} \mathbf{H} = 4\pi \mathbf{j}, \quad (1.5)$$

$$\operatorname{curl} \mathbf{E} = -\mu_e \frac{\partial \mathbf{H}}{\partial t}, \quad (1.6)$$

and finally Ohm's law for the moving fluid

$$\mathbf{j} = \sigma(\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H}). \quad (1.7)$$

In equation (1.5) the displacement current is neglected; for steady motion, however, the equation is exact.

For steady motion, on taking the curl of equation (1.7) and using (1.6) and (1.5), there results

$$\nabla \times (\nabla \times \mathbf{H}) = 4\pi \sigma \mu_e \nabla \times (\mathbf{q} \times \mathbf{H}). \quad (1.8)$$

On expanding $\nabla \times (\nabla \times \mathbf{H})$ and $\nabla \times (\mathbf{q} \times \mathbf{H})$ and using equations (1.1) and (1.4), equation (1.8) becomes

$$\lambda \nabla^2 \mathbf{H} = (\mathbf{q} \cdot \nabla) \mathbf{H} - (\mathbf{H} \cdot \nabla) \mathbf{q} \quad (1.9)$$

where $\lambda = 1/4\pi \sigma \mu_e$; λ is defined as the magnetic viscosity. Also on using equation (1.5) the equation of motion for steady flow becomes

$$\frac{1}{\rho} \nabla p + (\mathbf{q} \cdot \nabla) \mathbf{q} = \nu \nabla^2 \mathbf{q} + \frac{\mu_e}{4\pi \rho} (\nabla \times \mathbf{H}) \times \mathbf{H}. \quad (1.10)$$

In cylindrical polar coordinates equations (1.9), (1.10), in component form, and equations (1.1), (1.4), give the set of equations:

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}(ru_z) = 0, \quad (1.11)$$

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \right] + \frac{\mu_e}{4\pi \rho} H_z \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right), \quad (1.12)$$

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{\partial^2 u_z}{\partial z^2} \right] + \frac{\mu_e}{4\pi \rho} H_r \left(\frac{\partial H_z}{\partial r} - \frac{\partial H_r}{\partial z} \right), \quad (1.13)$$

$$\frac{\partial}{\partial r}(rH_r) + \frac{\partial}{\partial z}(rH_z) = 0, \quad (1.14)$$

$$u_r \frac{\partial H_r}{\partial r} + u_z \frac{\partial H_r}{\partial z} - H_r \frac{\partial u_r}{\partial r} - H_z \frac{\partial u_r}{\partial z} = \lambda \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_r}{\partial r} \right) + \frac{\partial^2 H_r}{\partial z^2} - \frac{H_r}{r^2} \right], \quad (1.15)$$

$$u_r \frac{\partial H_z}{\partial r} + u_z \frac{\partial H_z}{\partial z} - H_r \frac{\partial u_z}{\partial r} - H_z \frac{\partial u_z}{\partial z} = \lambda \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_z}{\partial r} \right) + \frac{\partial^2 H_z}{\partial z^2} \right]. \quad (1.16)$$

Finally, the energy equation (1.3), which describes the transport of thermal energy in the presence of a magnetic field is, for axially symmetric steady motion,

$$\rho c_p \left(u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \Phi + \frac{1}{\sigma} \mathbf{j}^2, \quad (1.17)$$

where Φ is the viscous dissipation function, and has the value

$$\Phi = 2\mu \left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 + \frac{u_r^2}{r^2} + \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)^2 \right]. \quad (1.18)$$

The term $(1/\sigma)\mathbf{j}^2$ is the Joule dissipation, and

$$\frac{1}{\sigma} \mathbf{j}^2 = \frac{1}{\sigma} j_\phi^2 = \frac{1}{16\pi^2\sigma} \left(\frac{\partial H_z}{\partial r} - \frac{\partial H_r}{\partial z} \right)^2. \quad (1.19)$$

2. Similarity solution of the equations, and boundary conditions

Consider now the magnetohydrodynamic equations (1.11) to (1.16). The boundary conditions appropriate to the flow under consideration are

$$\left. \begin{aligned} u_r = u_z = 0 & \quad \text{for } z = 0 \\ u_r \rightarrow ar & \quad \text{as } z \rightarrow \infty \quad \text{for all } r \end{aligned} \right\}, \quad (2.1)$$

where a is constant. Following Homann (4), who discussed the similar solution for the field free case, we introduce the new variable

$$\zeta = (a/\nu)^{1/2} z, \quad (2.2)$$

and the velocity components

$$\mathbf{q} = (u_r, 0, u_z) = [raf'(\zeta), 0, -2(av)^{1/2}f(\zeta)]. \quad (2.3)$$

When these substitutions are made in the basic equations of motion (1.11) to (1.16), a similar solution for axially symmetric stagnation point flow in the presence of a magnetic field is found to be possible if the law of deformation of H in component form is taken as

$$\mathbf{H} = (H_r, 0, H_z) = \left(-H_w \left(\frac{a}{\nu} \right)^{1/2} rh'(\zeta), 0, 2H_w h(\zeta) \right), \quad (2.4)$$

where H_w is a representative magnetic intensity. The above expressions for \mathbf{q} and \mathbf{H} satisfy identically equations (1.11) and (1.14), and equations

(1.12) and (1.13) become:

$$ra^2(f''' + 2ff'' - (f')^2 - 2\alpha hh'') - \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (2.5)$$

$$va(4ff' + 2f'') + \alpha a^2 rh'h'' + \frac{1}{\rho} \frac{\partial p}{\partial \zeta} = 0, \quad (2.6)$$

where $\alpha = \mu_e H_w^2 / 4\pi\rho va$. The pressure may be eliminated between equations (2.5) and (2.6); the resulting equation can be integrated once with respect to ζ , and we obtain an ordinary differential equation defining f as follows:

$$f''' + 2ff'' - (f')^2 + \alpha[(h')^2 - 2hh''] + C = 0, \quad (2.7)$$

where C is a constant of integration. The boundary conditions on f are

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \quad (2.8)$$

The corresponding equations for the magnetic field, namely (1.15) and (1.16), are now

$$\beta h''' = hf'' - fh'', \quad (2.9)$$

and

$$\beta h'' = hf' - fh', \quad (2.10)$$

where β is the ratio of the magnetic viscosity to twice the kinematic viscosity. Since equation (2.10) is a first integral of equation (2.9) the validity of the similar solution assumed in expressions (2.3) and (2.4) is established. It remains now to state the boundary conditions for \mathbf{H} . The similar solution for \mathbf{H} implies, from the usual condition on \mathbf{B} and \mathbf{H} at the solid interface, that the field in the solid must satisfy the boundary conditions

$$\mathbf{H} = (H_r, H_\theta, H_z) = \left(-H_w \left(\frac{a}{v} \right)^{\frac{1}{2}} rh'(0), 0, 2\gamma H_w h(0) \right), \quad (2.11)$$

at $z = 0$ for all r , where γ is the ratio of the magnetic permeability of the fluid to that of the solid. If we suppose the solid is non-conducting the magnetic field inside the solid is determined by $\mathbf{H} = \nabla\phi$, where ϕ satisfies the Laplace equation $\nabla^2\phi = 0$. If now we impose the restriction that the magnetic field is to remain finite as $z \rightarrow -\infty$ then in the solid the only possible solution of the form required is

$$H_r = H_\theta = 0, \quad \text{and} \quad H_z = 2\gamma H_w h(0). \quad (2.12)$$

Thus for the fluid at the solid interface the boundary conditions on \mathbf{H} are $\mathbf{H} = (H_r, H_\theta, H_z) = (0, 0, 2H_w)$ and hence

$$h(0) = 1, \quad \text{and} \quad h'(0) = 0. \quad (2.13)$$

For a physical interpretation of the above situation in which the magnetic field in the solid is uniform and in the z -direction we might consider the whole assembly to be placed inside a large solenoid which provides an applied magnetic field in the solid, so that the field is approximately

uniform in the neighbourhood of the axis $r = 0$, and acts in the z -direction; the resultant field in the fluid is the sum of this field and that due to the motion of the fluid. A model corresponding approximately to the solution discussed here is that in which the whole assembly becomes infinitely large.†

Returning to equation (2.7) the undetermined constant C may be specified by examination of the solution of equations (2.7) and (2.10) when ζ is large. Since, for large ζ , $f'(\zeta) \sim 1$ and $f(\zeta) \sim \zeta + A$, then from equation (2.10), $h'(\zeta) \sim B$ and $h(\zeta) \sim B(\zeta + A)$, where A and B are further constants of integration. Furthermore $h''(\zeta) \sim 0$ and if we seek a solution for f in which $f''(\zeta)$ and all higher derivatives vanish identically, then

$$C = 1 - \alpha B^2 = 1 - \alpha(h'(\zeta))_{\zeta \rightarrow \infty}^2. \quad (2.14)$$

Hence, to recapitulate, we need to solve the following coupled system of ordinary differential equations:

$$f''' + 2ff'' + 1 - (f')^2 + \alpha[(h')^2 - 2hh'' - (h')_{\zeta \rightarrow \infty}^2] = 0, \quad (2.15)$$

$$\beta h'' + fh' - f'h = 0, \quad (2.16)$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1, \quad h(0) = 1, \quad h'(0) = 0, \quad (2.17)$$

where $\alpha = \mu_e H_w^2 / 4\pi\rho\nu a$ and $\beta = \lambda/2\nu = (8\pi\sigma\mu_e\nu)^{-1}$. We note that when $\alpha = 0$, i.e. $H_w = 0$, equations (2.15) and (2.16) uncouple and equation (2.15) reduces to the field free case discussed by Homann (4) and Frössling (5).

3. Series solution of the ordinary differential equations (2.15) and (2.16)

The numerical solution of equations (2.15) and (2.16) subject to the boundary conditions (2.17) is complicated by non-linearity, coupling, and the fact that for most practical purposes β is very large, say of the order 10^5 or 10^6 ; also equation (2.15) is of the boundary value type and (2.16) is of the initial value type. The complication of non-linearity and coupling is reduced by expressing f and h as a series in powers of the magnetic parameter $m = \alpha/\beta = \mu_e^2 H_w^2 2\sigma/\rho a$, for fixed β ; i.e.

$$f(\zeta) = \sum_{r=0}^{\infty} m^r f_r(\zeta), \quad h(\zeta) = \sum_{r=0}^{\infty} m^r h_r(\zeta). \quad (3.1)$$

On substituting the series (3.1) into equations (2.15) and (2.16) and equating coefficients of like powers of m , there results a set of differential

† The authors are indebted to a referee for suggesting this model.

equations, of which the first three are:

$$f_0''' + 2f_0 f_0'' - (f_0')^2 + 1 = 0, \quad (3.2)$$

$$\beta h_0'' + f_0 h_0' - f_0' h_0 = 0, \quad (3.3)$$

$$f_1''' + 2f_0 f_1'' - 2f_0' f_1' + 2f_0'' f_1 = \beta [2h_0 h_0'' - (h_0')^2 + (h_0')_{\zeta=\infty}^2], \quad (3.4)$$

subject to the boundary conditions

$$f_0(0) = f_0'(0) = 0, \quad f_0'(\infty) = 1, \quad (3.2a)$$

$$h_0(0) = 1, \quad h_0'(0) = 0, \quad (3.3a)$$

$$f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = 0. \quad (3.4a)$$

Consider first equation (3.2). From the numerical solution for f_0 , obtained by Frössling (5), we observe that for $\zeta \geq \zeta_0$ (where $\zeta_0 = 3.7$) we have $f_0 = \zeta + A$. Thus there are two regions of interest: (a) an 'inner region' for $\zeta \leq \zeta_0$ in which viscosity is important, where f_0 and its higher derivatives vary rapidly from their initial values at $\zeta = 0$ to their 'asymptotic values' at $\zeta = \zeta_0$, and (b) an 'outer region' of flow in which the viscosity is unimportant, and where $f_0 = \zeta + A$ for $\zeta_0 < \zeta < \infty$.

Equation (3.3) and the boundary conditions (3.3a) define an initial value problem for h_0 . It is convenient to transform equation (3.3) by introducing the new dependent variable \bar{h}_0 , defined by

$$h_0 = 1 + \frac{\bar{h}_0}{\beta}, \quad (3.5)$$

so that the equation and boundary conditions become

$$\bar{h}_0'' = f_0' + \frac{1}{\beta} (\bar{h}_0 f_0' - f_0' \bar{h}_0), \quad \bar{h}_0(0) = \bar{h}_0'(0) = 0. \quad (3.6)$$

The numerical solution of this equation may be obtained by step-by-step methods.† When $\zeta \geq \zeta_0$ equation (3.3) becomes

$$\beta H_0'' + (\zeta + A) H_0' - H_0 = 0, \quad (3.7)$$

where for clarity the outer solution for h_0 is denoted by H_0 . Hence

$$H_0 = b_0 \eta + c_0 \left(e^{-\frac{1}{2}\eta^2} - \eta \int_{\eta}^{\infty} e^{-\frac{1}{2}\eta'^2} d\eta' \right), \quad (3.8)$$

where $\sqrt{\beta}\eta = \zeta + A$, and b_0 and c_0 are constants of integration which may be found by joining the inner and outer solutions at $\zeta = \zeta_0$. The 'inner' and 'outer' solutions to h_0 defined by $\bar{h}_0(\zeta)$ and $H_0(\eta)$ respectively are now known to the same accuracy as the accepted tables of f_0 , and in this sense we can say \bar{h}_0 is 'exact'.

† Note that when $\beta = 10^6$ there is fortuitous cancellation inside the bracket on the right-hand side of equation (3.6). Hence a good approximation to \bar{h}_0 is $\int_0^{\zeta} f_0 d\zeta$ for $0 < \zeta < \zeta_0$.

The equation for f_1 is of the ordinary boundary value type and we can solve this equation using step-by-step methods, bearing in mind that we must consider the 'inner' and 'outer' solutions of f_1 corresponding to the 'inner' and 'outer' solutions of h_0 . If the operator $L(D)$ is defined by

$$L(D) = \left(\frac{d^3}{d\zeta^3} + 2f_0 \frac{d^2}{d\zeta^2} - 2f_0' \frac{d}{d\zeta} + 2f_0'' \right),$$

then the 'inner' solution for f_1 is obtained by solving

$$L(D)f_1 = b_0^2 + 2 \left(1 + \frac{h_0}{\beta} \right) h_0'' - \frac{(h_0')^2}{\beta}. \quad (3.9)$$

For $\zeta \leq \zeta_0$ the required solution of equation (3.9) may be written

$$f_1 = f_1^{(p)} + \gamma_1 f_1^{(c)}, \quad (3.10)$$

where $f_1^{(p)}$ is a particular integral and $f_1^{(c)}$ is a complementary function, each of which may be chosen to satisfy the boundary conditions on f_1 at $\zeta = 0$. The constant γ_1 remains to be determined from the boundary conditions at infinity.

When $\zeta \geq \zeta_0$, equation (3.4) becomes

$$f_1''' + 2(\zeta + A)f_1'' - 2f_1' = \beta(h_0')^2_{\zeta=\infty} + \beta(2h_0 h_0'' - (h_0')^2). \quad (3.11)$$

Since a numerical solution of this equation is necessary, it is convenient to make the change of variables

$$\zeta + A = \sqrt{\beta}\eta, \quad f_1(\zeta) = \sqrt{\beta}F_1(\eta), \quad (3.12)$$

so that the equation becomes

$$\frac{1}{\beta} F_1''' + 2(\eta F_1'' - F_1') = (H_0')^2_{\eta=\infty} + 2H_0 H_0'' - (H_0')^2 = t(\eta), \quad \text{say.} \quad (3.13)$$

The solution of this equation must be discussed in some detail, since the method applies to subsequent equations; the physical meaning of the approximations involved is considered later. The parameter β is very large, and it is reasonable to expect that a suitable approximate solution of the equation may be obtained by neglecting the term F_1'''/β , so that the equation becomes

$$2(\eta F_1'' - F_1') = t(\eta). \quad (3.13a)$$

The third-order equation (3.13) is thus replaced by the second-order equation (3.13a), so that one boundary condition must be abandoned; furthermore, the solutions of both equations must be compared. In the first place we can obtain a particular integral $F_1^{(p)}$ which satisfies equation (3.13a), and also equation (3.13) correct to terms $O(1/\beta)$.

The general solution of equation (3.13) is then

$$F_1 = a_1 \left(\eta^2 \int_0^\eta e^{-\beta\eta^2} d\eta + \frac{1}{2\beta} \int_0^\eta e^{-\beta\eta^2} d\eta + \frac{\eta}{2\beta} e^{-\beta\eta^2} \right) + a_2 \eta^2 + a_3 + F_1^{(p)}, \quad (3.14)$$

and that of equation (3.13 a)

$$F_1 = b_2 \eta^2 + b_3 + F_1^{(p)}. \quad (3.14 a)$$

These functions differ by the term in a_1 , and since we require to join the inner and outer solutions for f_1 at $\zeta = \zeta_0 = 3.7$ we consider the behaviour of this term for $\eta > \eta_0$, where η_0 is the corresponding value of η . Asymptotic expansions may then be used for the integral, and the coefficient of a_1 is found to be

$$\frac{1}{2} \left(\frac{\pi}{\beta} \right)^{\frac{1}{2}} \eta^2 + \frac{1}{4\beta} \left(\frac{\pi}{\beta} \right)^{\frac{1}{2}} + \frac{\eta}{2\beta} e^{-\beta\eta^2} + O \left(\frac{1}{\beta^2 \eta} e^{-\beta\eta^2} \right).$$

For the range of values of η under consideration, the first two terms alone in the above expression are significant (the exponential terms are of order e^{-9}), so that the solution (3.14) is, effectively, the same as (3.14 a). Since the particular integral $F_1^{(p)}$ is correct to order $1/\beta$, we may expect this accuracy in the evaluation of F_1 .

The boundary condition at infinity is satisfied by choosing b_2 so that

$$2b_2 \eta + \frac{d}{d\eta} F_1^{(p)} \rightarrow 0 \quad \text{as } \eta \rightarrow \infty,$$

and the value of γ_1 in (3.10) is then obtained by ensuring continuity in $df_1/d\zeta$ at $\zeta = \zeta_0$; b_3 is obtained by numerical integration. The results show that the term neglected in equation (3.13) is indeed small.

The higher order functions h_1 , f_2 , h_2 , and f_3 may now be obtained in a similar fashion, i.e. we calculate the 'inner' solutions defined by $f_r(\zeta)$ and

$$h_r(\zeta) = \delta_r^0 + \frac{1}{\beta} h_r(\zeta), \quad (3.15)$$

where δ_r^0 is the Kronecker delta, together with the 'outer' solutions defined by $F_r(\eta)$ and $H_r(\eta)$ respectively. We note that the 'inner' and 'outer' solutions for $f_r(\zeta)$ ($r \geq 1$) are thus known approximately to within terms of order $1/\beta$, since in general we have neglected the term $(1/\beta)F_r'''$ in the differential equations which describe the outer solution. As this term originates from the viscous terms in the original flow equations (1.12) and (1.13) the above approximation is equivalent to stating that the outer solution applies to a region in which the viscosity is unimportant. In the next section a physical interpretation of this approximation is discussed.

4. The magneto-viscous and magneto-inviscid layers

The analysis of the previous section reveals the existence of three régimes of flow. In the first place there is a layer of fluid near the wall, which may be described as the magneto-viscous layer, and this is of the conventional boundary layer type. That is, inertia terms and viscous forces are comparable in magnitude, and the thickness is of the order $(\nu/a)^{1/2}$ as in the field free case. However, the Lorentz force is important in this layer, and furthermore the pressure distribution throughout the layer and the velocity distribution near the edge of the layer are greatly influenced by the adjacent magneto-inviscid layer.

The magneto-inviscid layer plays the role of a buffer layer between the magneto-viscous layer and the potential flow régime in the free stream. In this layer viscosity is unimportant, but the Lorentz force is comparable in magnitude with the inertia terms. From section 3 we find that this layer has thickness of order $(8\pi\sigma\mu_e a)^{-1/2}$, so that if $\beta = O(10^6)$ the ratio of this thickness to that of the magneto-viscous layer is $O(10^3)$. Again, from section 3, the error introduced by considering this layer to be inviscid is seen to be of the order β^{-1} , and is thus extremely small in the case under consideration. Thus equations describing the field and the flow in this layer may be obtained by putting $\nu = 0$ in equations (1.11) to (1.16). If, further, we introduce for this layer the variable η defined in (3.12), and velocity and field components defined by

$$\mathbf{q} = [raF'(\eta), 0, -2a(8\pi\sigma\mu_e a)^{-1/2}F(\eta)], \quad (4.1)$$

$$\mathbf{H} = [-H_w(8\pi\sigma\mu_e a)^{1/2}rH'(\eta), 0, 2H_w H(\eta)], \quad (4.2)$$

we obtain ordinary differential equations for F and H by the process described in section 2 for f and h . Expansion in series as in section 3, namely,

$$F(\eta) = \sum_{r=0}^{\infty} m^r F_r(\eta), \quad H(\eta) = \sum_{r=0}^{\infty} m^r H_r(\eta), \quad (4.3)$$

then yields a set of equations for F_r and H_r .

Thus the solution described in section 3 is obtained by joining the solution in the magneto-viscous layer to that in the magneto-inviscid layer. Since the continuity of velocity and field components u_r , u_z , H_r , H_z must be ensured at the interface, the following conditions are obtained at $\eta_0 = (\zeta_0 + A)/\beta$:

$$\left. \begin{aligned} f_r(\zeta_0) &= \sqrt{\beta} F_r(\eta_0), & \delta_r^0 + \frac{1}{\beta} h_r(\zeta_0) &= H_r(\eta_0) \\ \left(\frac{df_r}{d\zeta}\right)_{\zeta=\zeta_0} &= \left(\frac{dF_r}{d\eta}\right)_{\eta=\eta_0}, & \left(\frac{dh_r}{d\zeta}\right)_{\zeta=\zeta_0} &= \sqrt{\beta} \left(\frac{dH_r}{d\eta}\right)_{\eta=\eta_0} \end{aligned} \right\}. \quad (4.4)$$

Above the magneto-inviscid layer lies the free stream; this régime of

flow is included in the equations for the magneto-inviscid layer, since it is represented by the solution for large values of η , but it is convenient to think of it physically as a separate layer of flow as in boundary layer theory.

Tables 1-4 give the functions $f_{r+1}(\zeta)$, $F_{r+1}(\eta)$, $h_r(\zeta)$, and $H_r(\eta)$ for $r = 0, 1$, and 2 , together with their associated derivatives, when $\beta = 10^6$.

5. Solution of the energy equation when the wall is thermally insulated

The partial differential equation which describes the transport of thermal energy for axially symmetric flow is†

$$\rho c_p \left(u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \mu \left[2 \left(\frac{\partial u_r}{\partial r} \right)^2 + 2 \left(\frac{\partial u_z}{\partial z} \right)^2 + \frac{2u_z^2}{r^2} + \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)^2 \right] + \frac{1}{16\pi^2\sigma} \left(\frac{\partial H_z}{\partial r} - \frac{\partial H_r}{\partial z} \right)^2, \quad (5.1)$$

where the second and third terms on the right-hand side of equation (5.1) represent the viscous and Joule dissipation respectively. We shall consider the case of a thermally insulated wall, i.e.

$$\left. \begin{aligned} \text{at } z = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{for all } r \\ \text{and } T \rightarrow T_\infty \quad \text{as } z \rightarrow \infty \end{aligned} \right\}, \quad (5.2)$$

where T_∞ is the bulk temperature of the fluid in the main stream.

Equation (5.1) may be analysed as follows. First we determine the conditions for which a similar solution exists as in section 2, and then proceed to calculate the outer solution describing the local temperature distribution correct to within quantities of order β^{-1} as in section 3. Thus we find on examination of equation (5.1) that for the transport of thermal energy it is sufficient to consider:

(A) a magneto-viscous thermal layer in which the convection, conduction, viscous, and Joule dissipation terms of (5.1) are important. Furthermore we neglect the first three terms of the viscous dissipation function and also the radial conduction, all of which are small in comparison with terms of the same order as the axial conduction.

(B) a magneto-inviscid thermal layer in which axial and radial conduction are negligible. Since in this region $\nu \equiv 0$, then only convection and Joule dissipation contribute to the transport of thermal energy.

† Note that equation (5.1) has limited application since $(T - T_\infty)/T_\infty$ must be small.

A. The magneto-viscous thermal layer

We introduce a dimensionless temperature distribution

$$\theta(\xi) = \frac{c_p(T - T_x)}{a^2 r^2}, \quad (5.3)$$

and substitution in equation (5.1) gives the equation

$$\frac{1}{P} \theta'' + 2(f\theta' - f'\theta) + (f'')^2 + 2m\beta^2(h'')^2 = 0, \quad (5.4)$$

where P is the Prandtl number. One boundary condition is

$$\theta'(0) = 0, \quad (5.5)$$

and the remaining condition must be obtained by ensuring continuity in the temperature at the interface between the magneto-viscous and magneto-inviscid thermal layers.

For small m we write

$$\theta(\xi) = P \sum_{r=0}^{\infty} m^r g_r(\xi), \quad (5.6)$$

and on using the expressions (3.1), (3.15) in (5.4), and equating the coefficients of the same powers of m , we have

$$g_0'' + 2P(f_0 g_0' - f_0' g_0) = -(f_0'')^2, \quad (5.7)$$

$$g_1'' + 2P(f_0 g_1' - f_0' g_1) = 2P(f_1' g_0 - f_1 g_0') - 2f_0 f_1'' - (f_0'')^2, \quad (5.8)$$

and so on. The boundary conditions for the g_r are

$$g_r'(0) = 0, \quad \text{all } r. \quad (5.9)$$

The remaining set of boundary conditions are introduced in the discussion of the magneto-inviscid thermal layer.

B. The magneto-inviscid thermal layer

In the magneto-inviscid thermal layer the mechanism for energy transport consists of convection and Joule dissipation, i.e.

$$\rho c_p \left(u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z} \right) = \frac{1}{16\pi^2 \sigma} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right)^2. \quad (5.10)$$

With the introduction of the new variable η defined in (3.12) together with velocity and magnetic field components, as defined by (4.1) and (4.2), and also the dimensionless local temperature distribution

$$\Theta(\eta) = \frac{c_p(T - T_x)}{a^2 r^2}, \quad (5.11)$$

we obtain

$$F'\Theta - F\Theta' = m(H'')^2, \quad (5.12)$$

to be solved subject to the boundary condition

$$\Theta(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (5.13)$$

Since for the field free solution $\Theta(\eta) \equiv 0$ when the Prandtl number is of order unity (for gases), we may write for small m

$$\Theta(\eta) = \sum_{r=1}^{\infty} m^r G_r(\eta). \quad (5.14)$$

Substitution of the series (5.14) and (4.3) into equation (5.12) gives the equations

$$\eta G'_1 - G_1 = -(H_0'')^2, \quad (5.15)$$

$$\eta G'_2 - G_2 = F_1' G_1 - F_1 G_1' - 2H_0' H_1', \quad (5.16)$$

and so on, where $G_r(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$.

Proceeding as in section 3 we now calculate a particular integral and a complementary function for $g_r(\zeta)$ in the interval $0 \leq \zeta \leq \zeta_0$. The appropriate combination is then determined from our knowledge of the particular integral $G_r(\eta)$. (The complementary function of $G_r(\eta)$ is unacceptable, since $G_r(\infty) = 0$.) In effect since the temperature must be continuous at the interface between the magneto-viscous and magneto-inviscid layers, then

$$Pg_r(\zeta_0) = G_r(\eta_0) \quad (5.17)$$

for $r = 1, 2, 3, \dots$

In Tables 5 and 6 the functions $g_r(\zeta)$ and $G_r(\eta)/P$ for $r = 0, 1, 2$, and 3, together with their first derivatives, are given in tabular form. The Prandtl number was taken to be 0.72.

6. Results and discussion

From Tables 1-6 quantitative information may now be deduced for flow characteristics such as the local shear stress and local temperature at the wall; the results are relevant only to the special case $\beta = 10^6$ and $P = 0.72$.

Consider first the local shear stress at the wall defined by

$$\tau_w = \mu \left(\frac{\partial u_r}{\partial z} \right)_{z=0} = \mu \left(\frac{a}{v} \right)^{\frac{1}{2}} arf''(0). \quad (6.1)$$

If τ_0 is the field free local shear stress at the wall, we obtain

$$\frac{\tau_w}{\tau_0} = \frac{1}{f_0''(0)} \{ f_0''(0) + m f_1''(0) + m^2 f_2''(0) + m^3 f_3''(0) + \dots \}, \quad (6.2)$$

where $f_0''(0) = 1.311_9$, and from Table 1,

$$f_1''(0) = -2.765_5, \quad f_2''(0) = 0.368_8, \quad f_3''(0) = -1.161.$$

The ratio τ_w/τ_0 is given graphically in Fig. 2 for various values of m . Owing to the truncated form of the calculated series (6.2) this ratio is known with certainty only for $0 \leq m \leq 0.2$. Thus for $m = 0.2$ there is a 41 per cent reduction in the local wall shear stress. However, it is

instructive to observe that the local shear stress appears to be zero for a value of m near to 0.46. When the field is as strong as this, the resulting type of flow pattern is not clear. Tentatively one might expect that when $\tau_w = 0$ a region of stationary fluid exists near the wall. The possibility of a solution existing for $\tau_w < 0$ (i.e. m greater than the estimated value 0.46) in which we should have a radial inflow along the wall seems to be unlikely. Obviously the need for further computations in the neighbourhood of $m = 0.4$ is indicated.

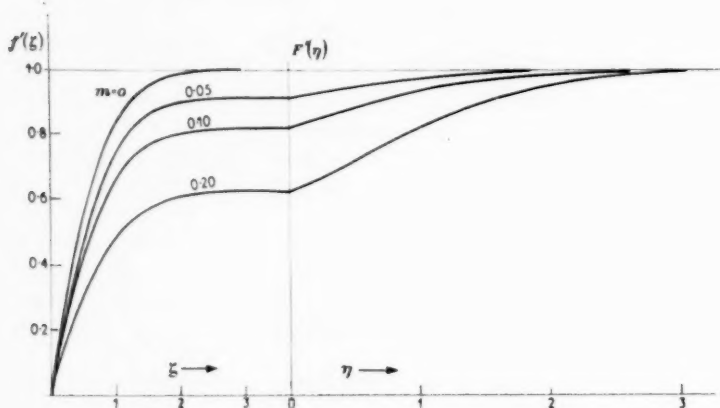


FIG. 1. Dimensionless radial velocity profiles, $u(r, z)/u(r, \infty) = f'(\xi)$, $F'(\eta)$, for various values of m , when $\beta = 10^6$.

The dimensionless radial velocity profiles $f'(\xi)$ for the magneto-viscous layer and $F'(\eta)$ for the magneto-inviscid layer are given graphically in Fig. 1 for $m = 0, 0.05, 0.10$, and 0.20 . Physically the influence of the magnetic field on shear stress and velocity profiles may be described as follows. As the radial velocity in the magneto-viscous layer increases from zero at the wall the induced current increases and consequently the Lorentz force $\mu_e(\mathbf{j} \times \mathbf{H})$ grows in strength. Since this force has a radial component which opposes the radial flow, the radial velocity will be retarded and thus reduced in magnitude as shown in Fig. 1. The radial velocity component will then increase slowly through the magneto-inviscid layer to the incident value in the layer of potential flow, as the Lorentz force decays slowly to zero. If in the magneto-viscous layer the magnitude of u_r is reduced, then so will be the local shear stress at the wall.

The effect of the magnetic field on the local temperature distribution at the wall, known as the eigentemperature, may be seen on examining

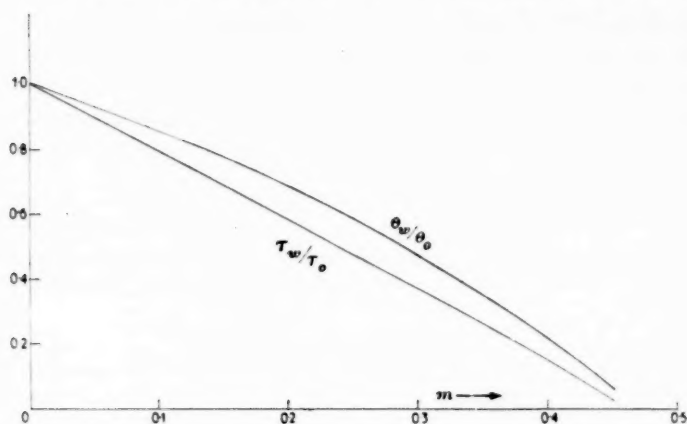


FIG. 2. Shear stress and eigentemperature ratios versus m when $\beta = 10^6$ and $P = 0.72$.

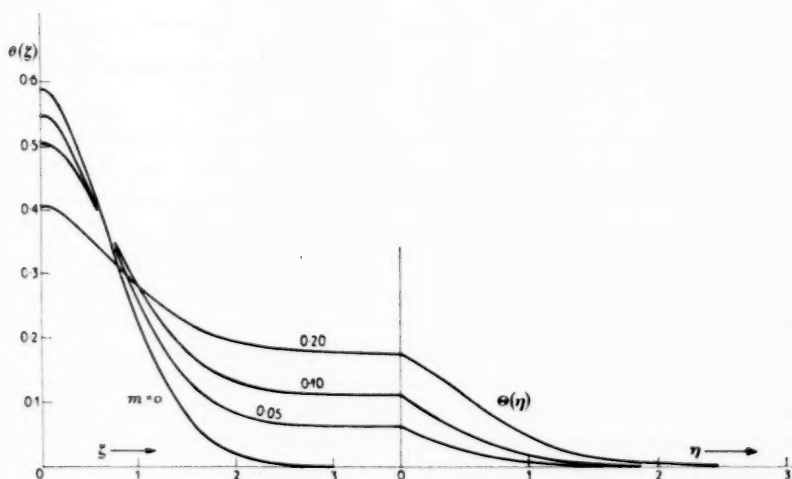


FIG. 3. Dimensionless local temperature profiles, $c_p(T - T_\infty)/a^2r^2 = \theta(\xi)$, $\Theta(\eta)$, for various values of m , when $\beta = 10^6$ and $P = 0.72$.

the ratio of $\theta_w = c_p(T_w - T_\infty)/a^2r^2$ with magnetic field to $\theta_w = \theta_0$ without the magnetic field. Thus

$$\frac{\theta_w}{\theta_0} = \frac{1}{g_0(0)} \{g_0(0) + mg_1(0) + m^2g_2(0) + m^3g_3(0) + \dots\}, \quad (6.3)$$

where from Table 5:

$$g_0(0) = 0.588_6, \quad g_1(0) = -0.730, \quad g_2(0) = -0.758_5, \quad g_3(0) = -0.795.$$

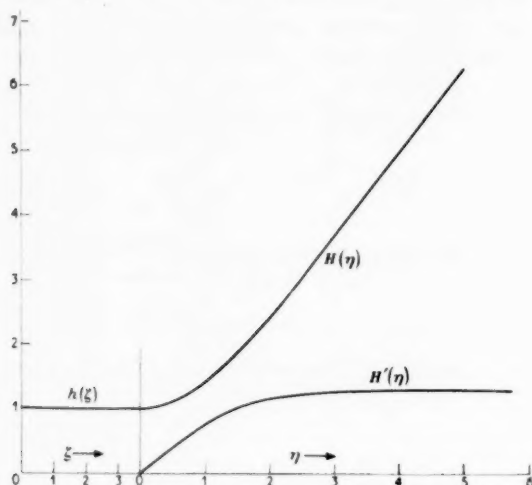


FIG. 4. Representative dimensionless magnetic field profiles (see expressions (2.4) and (4.2)), when $\beta = 10^6$ and $0 \leq m \leq 0.2$.

The ratio θ_w/θ_0 is given graphically in Fig. 2 and representative dimensionless thermal profiles (θ, Θ) are given in Fig. 3 for $m = 0, 0.05, 0.10$, and 0.20 . We note that when $m = 0.2$ there is a 31 per cent decrease in eigentemperature. Furthermore, assuming that equation (6.3) is valid for $m > 0.2$ we observe from Fig. 2 that the eigentemperature will be zero for m slightly in excess of 0.46 . The question of whether or not such a situation is possible remains unanswered. However, the reason for the reduction in eigentemperature θ_w and the local temperature distribution throughout the magneto-viscous and inviscid thermal layers can be seen on examination of the energy equation (5.1). In the magneto-viscous thermal layer near the wall the Joule dissipation will be negligible but there is a considerable reduction in the viscous dissipation, i.e. the term $\mu(\partial u_r/\partial z)^2$. Since the energy equation (5.1) is linear in T it follows that the net effect of the magnetic field is to reduce the local temperature distribution. In the magneto-inviscid thermal layer the balance of thermal energy is controlled by the Joule dissipation and as this decays slowly to zero in consequence T tends slowly to T_∞ .

The magnetic field vectors are not without interest. In the magneto-viscous layer the axial component H_z is for all practical purposes constant.

However, slow variations in both H_r and H_z occur in the magneto-inviscid layer. Using Tables 3 and 4 we find that to within a few per cent the components H_z and H_r are insensitive to m for $0 \leq m \leq 0.2$. The dimensionless magnetic profiles $h(\zeta)$, $H(\eta)$, and $h'(\zeta)$, $H'(\eta)$, defined by equations (2.4) and (4.2), are shown in Fig. 4 by the representative curves for $m = 0.1$.

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TABLE I

ξ	f_1	f'_1	f''_1	f_2	f'_2	f''_2	f_3	f'_3	f''_3
0.0	-0.000	-0.000	-2.765 ₈	0.000	0.000	0.368 ₈	-0.000	-0.000	-1.161
0.1	-0.013 ₆	-0.268 ₃	-2.596	0.002 ₀	0.039 ₈	0.412 ₇	-0.005 ₈	-0.109 ₆	-1.030
0.2	-0.053 ₁	-0.518 ₄	-2.402	0.008 ₀	0.080 ₈	0.405 ₃	-0.021 ₃	-0.205 ₈	-0.890 ₃
0.3	-0.116 ₆	-0.718 ₄	-2.189	0.018 ₀	0.119 ₁	0.352 ₆	-0.046 ₆	-0.287 ₄	-0.745 ₁
0.4	-0.201 ₉	-0.955 ₈	-1.960	0.031 ₅	0.150 ₁	0.262 ₆	-0.078 ₅	-0.354 ₆	-0.598 ₁
0.5	-0.306 ₉	-1.140	-1.721	0.047 ₇	0.170 ₇	0.145 ₄	-0.116 ₇	-0.407 ₃	-0.455 ₉
0.6	-0.429 ₁	-1.300	-1.478 ₈	0.065 ₃	0.178 ₇	0.013 ₆	-0.159 ₄	-0.446 ₆	-0.327 ₁
0.7	-0.566 ₀	-1.435 ₈	-1.239	0.083 ₀	0.173 ₃	-0.120 ₈	-0.205 ₅	-0.473 ₃	-0.220 ₅
0.8	-0.715 ₈	-1.548	-1.009 ₈	0.099 ₈	0.154 ₉	-0.244 ₈	-0.253 ₈	-0.491 ₃	-0.143 ₈
0.9	-0.874 ₈	-1.638	-0.795 ₈	0.113 ₅	0.125 ₀	-0.348 ₀	-0.303 ₆	-0.503 ₁	-0.099 ₇
1.0	-1.042	-1.708	-0.602	0.124 ₈	0.086 ₂	-0.422 ₈	-0.354 ₃	-0.512 ₃	-0.090 ₀
1.1	-1.216	-1.759	-0.433 ₆	0.130 ₆	0.041 ₆	-0.464 ₂	-0.406 ₀	-0.522 ₁	-0.103 ₄
1.2	-1.394	-1.795	-0.291 ₅	0.132 ₄	-0.005 ₅	-0.472 ₄	-0.458 ₉	-0.534 ₉	-0.149 ₅
1.3	-1.574	-1.818 ₆	-0.176 ₃	0.129 ₆	-0.051 ₉	-0.450 ₄	-0.513 ₃	-0.552 ₃	-0.199 ₁
1.4	-1.757	-1.831 ₄	-0.087 ₃	0.122 ₁	-0.094 ₈	-0.404 ₁	-0.569 ₅	-0.574 ₇	-0.241 ₆
1.5	-1.940 ₆	-1.837	-0.022 ₃	0.110 ₇	-0.132 ₂	-0.341 ₂	-0.628 ₃	-0.601 ₃	-0.282 ₁
1.6	-2.124	-1.836 ₆	+0.022 ₀	0.095 ₉	-0.162 ₈	-0.266 ₉	-0.689 ₈	-0.630 ₅	-0.293 ₇
1.7	-2.307 ₈	-1.833	0.049 ₁	0.078 ₄	-0.186 ₁	-0.197 ₈	-0.754 ₄	-0.660 ₃	-0.293 ₃
1.8	-2.490 ₃	-1.827	0.062 ₈	0.058 ₉	-0.202 ₅	-0.131 ₀	-0.821 ₈	-0.688 ₄	-0.267 ₂
1.9	-2.673	-1.821	0.066 ₇	0.038 ₁	-0.212 ₇	-0.073 ₉	-0.891 ₉	-0.713 ₁	-0.224 ₅
2.0	-2.855	-1.814	0.064 ₀	0.016 ₆	-0.217 ₇	-0.028 ₉	-0.964 ₃	-0.733 ₀	-0.171 ₅
2.1	-3.036	-1.808	0.057 ₃	-0.005 ₃	-0.218 ₉	0.003 ₆	-1.038 ₄	-0.747 ₃	-0.114 ₃
2.2	-3.216 ₆	-1.803	0.048 ₇	-0.027 ₁	-0.217 ₄	0.024 ₄	-1.113 ₆	-0.756 ₁	-0.060 ₉
2.3	-3.396 ₆	-1.798	0.039 ₇	-0.048 ₇	-0.214 ₃	0.035 ₃	-1.180 ₄	-0.759 ₆	-0.014 ₃
2.4	-3.576	-1.795	0.031 ₁	-0.070 ₉	-0.210 ₆	0.038 ₆	-1.265 ₄	-0.759 ₃	0.022 ₂
2.5	-3.755 ₈	-1.792	0.023 ₇	-0.090 ₈	-0.206 ₆	0.036 ₆	-1.341 ₄	-0.755 ₇	0.047 ₆
2.6	-3.934 ₈	-1.790	0.017 ₄	-0.111 ₃	-0.203 ₃	0.031 ₃	-1.416 ₅	-0.750 ₁	0.062 ₆
2.7	-4.114	-1.788 ₃	0.012 ₅	-0.131 ₆	-0.200 ₆	0.024 ₆	-1.491 ₄	-0.743 ₅	0.068 ₆
2.8	-4.293	-1.787 ₅	0.008 ₇	-0.151 ₅	-0.198 ₃	0.017 ₃	-1.565 ₄	-0.736 ₈	0.067 ₃
2.9	-4.471	-1.787	0.005 ₉	-0.171 ₃	-0.197 ₁	0.010 ₇	-1.638 ₇	-0.730 ₁	0.062 ₇
3.0	-4.650	-1.786	0.003 ₉	-0.190 ₉	-0.196 ₃	0.004 ₉	-1.711 ₄	-0.724 ₂	0.055 ₀
3.1	-4.828 ₈	-1.786	0.002 ₅	-0.210 ₆	-0.196 ₁	0.000 ₃	-1.783 ₆	-0.719 ₁	0.046 ₁
3.2	-5.007	-1.786	0.001 ₆	-0.230 ₃	-0.196 ₄	-0.003 ₆	-1.855 ₈	-0.714 ₉	0.038 ₀
3.3	-5.186	-1.786	0.001 ₀	-0.249 ₈	-0.196 ₇	-0.006 ₁	-1.926 ₆	-0.711 ₅	0.030 ₅
3.4	-5.364	-1.786	0.000 ₈	-0.269 ₃	-0.197 ₈	-0.008 ₁	-1.997 ₆	-0.708 ₇	0.024 ₂
3.5	-5.543	-1.785 ₈	0.000 ₄	-0.289 ₃	-0.198 ₄	-0.009 ₅	-2.068 ₄	-0.706 ₆	0.019 ₀
3.6	-5.721 ₈	-1.785 ₆	0.000 ₂	-0.309 ₂	-0.199 ₄	-0.010 ₄	-2.138 ₉	-0.704 ₈	0.015 ₈
3.7	-5.900	-1.785 ₈	0.000 ₁	-0.329 ₂	-0.200 ₄	-0.011 ₀	-2.209 ₃	-0.703 ₆	0.012 ₄

TABLE 2

ζ	h_0	h'_0	h_1	h'_1	h_2	h'_2
0.0	0.000	0.000	-0.000	-0.000	0.000	0.000
0.1	0.000 ₂	0.000 ₄	-0.000 ₃	-0.013 ₆	0.000 ₁	0.001 ₄
0.2	0.001 ₇	0.002 ₉	-0.003 ₆	-0.053 ₁	0.000 ₅	0.008 ₀
0.3	0.005 ₆	0.005 ₄	-0.011 ₁	-0.116 ₈	0.001 ₈	0.018 ₀
0.4	0.012 ₉	0.009 ₃	-0.027 ₆	-0.201 ₉	0.004 ₃	0.031 ₅
0.5	0.024 ₇	0.014 _{3₂}	-0.052 ₉	-0.306 ₉	0.008 ₇	0.047 ₇
0.6	0.041 ₄	0.020 ₃	-0.080 ₆	-0.420 ₁	0.013 ₆	0.065 ₃
0.7	0.065 ₀	0.026 _{4₇}	-0.139 ₄	-0.566 ₀	0.021 ₂	0.083 ₀
0.8	0.095 ₀	0.0335 ₄	-0.220 _{3₂}	-0.715 ₄	0.030 ₄	0.099 ₅
0.9	0.132 ₃	0.0411 ₆	-0.282 ₆	-0.874 ₈	0.039 ₅	0.113 ₈
1.0	0.177 ₄	0.0492 ₄	-0.378 ₄	-1.042	0.051 ₄	0.124 ₆
1.1	0.230 ₉	0.0577 ₃	-0.491 ₉	-1.216	0.064 ₂	0.130 ₆
1.2	0.293 ₀	0.0665 ₄	-0.621 ₇	-1.393 ₃	0.077 ₃	0.132 ₄
1.3	0.364 ₁	0.0756 ₅	-0.770 ₁	-1.574 ₅	0.090 ₅	0.129 ₈
1.4	0.444 ₂	0.0849 ₃	-0.936 ₇	-1.757	0.103 ₁	0.122 ₁
1.5	0.534 ₀	0.0944 ₁	-1.121 ₅	-1.940 ₃	0.114 ₅	0.110 ₇
1.6	0.633 ₂	1.040	-1.325	-2.124	0.125 ₂	0.095 ₉
1.7	0.742 ₁	1.137 ₅	-1.546	-2.308	0.133 ₉	0.078 ₄
1.8	0.860 ₇	1.235 ₅	-1.786	-2.491	0.140 ₈	0.058 ₉
1.9	0.980 ₂	1.334	-2.044	-2.673	0.145 ₆	0.038 ₁
2.0	1.127 ₅	1.433	-2.321	-2.855	0.148 ₅	+0.016 ₆
2.1	1.276	1.532 ₅	-2.615	-3.036	0.149 ₀	-0.005 ₃
2.2	1.434	1.632	-2.928	-3.216 ₈	0.147 ₃	-0.027 ₁
2.3	1.602	1.732	-3.259	-3.396 ₈	0.143 ₅	-0.048 ₇
2.4	1.780	1.831	-3.607	-3.576	0.137 ₆	-0.070 ₀
2.5	1.968 ₅	1.931	-3.974	-3.755 ₆	0.129 ₆	-0.090 ₈
2.6	2.167	2.031	-4.358	-3.934 ₈	0.119 ₅	-0.111 ₃
2.7	2.375	2.131	-4.761	-4.113 ₈	0.107 ₃	-0.131 ₅
2.8	2.593	2.231	-5.181	-4.292	0.093 ₂	-0.151 ₅
2.9	2.821	2.331	-5.619	-4.471	0.077 ₀	-0.171 ₂
3.0	3.059	2.431	-6.075	-4.650	0.058 ₉	-0.190 ₆
3.1	3.307	2.531	-6.549	-4.828	0.038 ₈	-0.210 ₆
3.2	3.563	2.631	-7.041	-5.007	+0.018 ₈	-0.230 ₁
3.3	3.833	2.731	-7.550	-5.185	-0.005 ₂	-0.249 ₀
3.4	4.115	2.831	-8.078	-5.364	-0.031 ₂	-0.269 ₅
3.5	4.400	2.931	-8.623	-5.542	-0.059 ₁	-0.289 ₃
3.6	4.698	3.031	-9.186	-5.720	-0.089 ₁	-0.309 ₂
3.7	5.006	3.131	-9.767	-5.899	-0.121 ₀	-0.329 ₂

TABLE 3

η	F_1	F'_1	F''_1	F_2	F'_2	F''_2	F_3	F'_3	F''_3
0.05	-0.087 ₄	-1.698	1.722	-0.017 ₈	-0.505 ₈	-5.048	-0.032 ₉	-0.670 ₇	-1.890
0.10	-0.170 ₂	-1.613	1.672	-0.048 ₄	-0.722 ₂	-3.780	-0.069 ₄	-0.790 ₆	-2.901
0.15	-0.248 ₆	-1.531	1.622 ₃	-0.086 ₂	-0.891 ₆	-3.023	-0.113 ₀	-0.949 ₁	-3.137
0.2	-0.323 ₂	-1.451	1.572 ₅	-0.137 ₈	-1.028	-2.474 ₅	-0.164 ₄	-1.107	-3.151
0.3	-0.460 ₇	-1.299	1.472 ₈	-0.251 ₀	-1.234	-1.681	-0.290 ₆	-1.413 ₅	-2.955
0.4	-0.583 ₂	-1.156	1.373	-0.381 ₇	-1.372	-1.101	-0.446 ₂	-1.694 ₅	-2.658
0.5	-0.692 ₂	-1.024	1.273 ₅	-0.523 ₆	-1.458	-0.642 ₅	-0.628 ₃	-1.944	-2.325 ₅
0.6	-0.788 ₅	-0.901 ₆	1.175	-0.671 ₉	-1.503	-0.265 ₅	-0.833 ₅	-2.159	-1.976
0.7	-0.873 ₀	-0.789 ₁	1.077	-0.823 ₀	-1.513	+0.049 ₁	-1.059 ₀	-2.339	-1.615 ₅
0.8	-0.946 ₆	-0.686 ₁	0.981 ₅	-0.973 ₇	-1.475	-2.311 ₇	-1.390 ₅	-2.482	-1.245 ₅
0.9	-1.015 ₃	-0.592 ₇	0.888 ₈	-1.121	-1.452 ₅	0.528 ₈	-1.554	-2.588	-0.866 ₃
1.0	-1.065 ₅	-0.508 ₂	0.797 ₈	-1.263 ₅	-1.390 ₅	-0.703 ₁	-1.817	-2.656	-0.491 ₇
1.1	-1.112 ₅	-0.433 ₀	0.711 ₀	-1.399	-1.313	0.838 ₂	-2.084	-2.686	-0.118 ₅
1.2	-1.152 ₅	-0.366 ₁	0.628 ₅	-1.526	-1.224	0.935 ₉	-2.353	-2.680	+0.242 ₅
1.3	-1.189 ₀	-0.307 ₁	0.551 ₁	-1.643	-1.127	0.999 ₂	-2.619	-2.638	0.582 ₉
1.4	-1.214 ₀	-0.255 ₇	0.479 ₀	-1.751	-1.025	1.031	-2.879	-2.564	0.894 ₅
1.5	-1.237	-0.211 ₁	0.412 ₆	-1.848	-0.921 ₉	1.034	-3.131	-2.461	1.168
1.6	-1.259 ₅	-0.172 ₉	0.352 ₂	-1.935	-0.819 ₃	1.013	-3.370 ₅	-2.332	1.398
1.7	-1.272	-0.140 ₅	0.297 ₉	-2.012	-0.719 ₀	0.972 ₀	-3.596 ₅	-2.183	1.579 ₅
1.8	-1.285	-0.113 ₂	0.249 ₅	-2.079 ₅	-0.625 ₅	0.914 ₆	-3.806 ₅	-2.018	1.709
1.9	-1.295	-0.090 ₃	0.207 ₁	-2.137 ₅	-0.537 ₃	0.845 ₅	-4.000	-1.842 ₅	1.787
2.0	-1.303	-0.071 ₆	0.170 ₉	-2.187	-0.456 ₅	0.769 ₂	-4.175	-1.662	1.815 ₅
2.1	-1.309	-0.056 ₉	0.138 ₆	-2.229	-0.383 ₅	0.688 ₇	-4.332	-1.481	1.798
2.2	-1.314	-0.043 ₇	0.111 ₆	-2.264	-0.318 ₀	0.607 ₂	-4.471	-1.304	1.741
2.3	-1.318	-0.033 ₇	0.089 ₁	-2.293	-0.262 ₁	0.527 ₈	-4.593	-1.134	1.651
2.4	-1.321	-0.025 ₉	0.070 ₄	-2.317	-0.213 ₁	0.452 ₂	-4.698	-0.974 ₉	1.536
2.5	-1.323	-0.019 ₅	0.055 ₁	-2.336	-0.171 ₄	0.382 ₄	-4.788	-0.827 ₅	1.403
2.6	-1.325	-0.014 ₆	0.042 ₇	-2.351	-0.130 ₄	0.319 ₀	-4.864	-0.694 ₀	1.260
2.7	-1.326	-0.010 ₆	0.032 ₇	-2.363	-0.107 ₄	0.262 ₇	-4.927 ₅	-0.575 ₃	1.112 ₅
2.8	-1.327	-0.008 ₀	0.024 ₅	-2.373	-0.083 ₆	0.213 ₆	-4.980	-0.471 ₄	0.967 ₀
2.9	-1.328 ₀	-0.005 ₉	0.018 ₇	-2.380	-0.064 ₅	0.171 ₄	-5.022	-0.381 ₄	0.827 ₇
3.0	-1.328 ₅	-0.004 ₂	0.013 ₉	-2.386	-0.049 ₁	0.135 ₄	-5.056 ₅	-0.305 ₆	0.698 ₁
3.1	-1.328 ₅	-0.003 ₀	0.010 ₂	-2.390	-0.037 ₁	0.106 ₅	-5.084	-0.241 ₇	0.580 ₂
3.2	-1.329 ₁	-0.002 ₅	0.007 ₄	-2.393	-0.027 ₅	0.082 ₄	-5.105	-0.189 ₁	0.475 ₅
3.3	-1.329 ₅	-0.001 ₅	0.005 ₄	-2.396	-0.020 ₄	0.063 ₀	-5.122	-0.140 ₂	0.384 ₄
3.4	-1.329 ₆	-0.001 ₀	0.003 ₉	-2.398	-0.014 ₉	0.047 ₆	-5.135	-0.111 ₇	0.306 ₀
3.5	-1.329 ₅	-0.0007	0.002 ₇	-2.399	-0.010 ₅	0.035 ₆	-5.144 ₅	-0.084 ₄	0.241 ₅
3.6	-1.329 ₅	-0.000 ₅	0.001 ₉	-2.400	-0.007 ₇	0.026 ₅	-5.152	-0.063 ₁	0.187 ₄
3.7	-1.329 ₀	-0.000 ₃	0.001 ₂	-2.400 ₅	-0.005 ₅	0.019 ₂	-5.157	-0.046 ₅	0.143 ₇
3.8	-1.329 ₀	-0.000 ₂	0.000 ₉	-2.401	-0.003 ₆	0.013 ₃	-5.161	-0.034 ₁	0.108 ₅
3.9	-1.329 ₆	-0.000 ₂	0.000 ₆	-2.401 ₅	-0.002 ₇	0.009 ₉	-5.164	-0.024 ₆	0.081 ₃
4.0	-1.329 ₆	-0.000 ₁	0.000 ₄	-2.402	-0.001 ₆	0.007 ₀	-5.166	-0.017 ₆	0.060 ₂
4.1	-1.329 ₆	-0.000 ₁	0.000 ₂	-2.402	-0.001 ₂	0.004 ₉	-5.168	-0.012 ₄	0.043 ₅
4.2	-1.329 ₇	-0.000	0.000 ₂	-2.402	-0.000 ₅	0.003 ₄	-5.169	-0.008 ₇	0.031 ₅
4.3				-2.402	-0.000 ₅	0.002 ₅	-5.169 ₅	-0.006 ₀	0.022 ₄
4.4				-2.402	-0.000 ₃	0.001 ₅	-5.170	-0.004 ₁	0.015 ₅
4.5				-2.402	-0.000 ₂	0.001 ₀	-5.170 ₅	-0.002 ₅	0.011 ₅
4.6				-2.402	-0.000 ₁	0.000 ₇	-5.171	-0.001 ₅	0.007 ₅
4.7							-5.171	-0.001 ₅	0.005 ₁
4.8								-0.000 ₅	0.003 ₅
4.9								-0.000 ₅	0.002 ₅
5.0								-0.000 ₄	0.001 ₅

TABLE 4

η	H_0	H'_0	H_1	H'_1	H_2	H'_2
0.05	1.001	0.050 ₀	-0.002 ₃	-0.087 ₃	-0.000 ₄	-0.017 ₃
0.10	1.005	0.099 ₀	-0.008 ₇	-0.169 ₀	-0.002 ₆	-0.048 ₈
0.15	1.011	0.149 ₀	-0.019 ₁	-0.246 ₀	-0.005 ₄	-0.089 ₀
0.2	1.020	0.198 ₇	-0.033 ₃	-0.318 ₀	-0.011 ₁	-0.139 ₃
0.3	1.045	0.295 ₀	-0.071 ₇	-0.446 ₁	-0.030 ₇	-0.257 ₁
0.4	1.079	0.389 ₀	-0.121 ₇	-0.550 ₁	-0.063 ₁	-0.392 ₇
0.5	1.122 ₈	0.479 ₀	-0.180 ₀	-0.630 ₄	-0.103 ₀	-0.538 ₁
0.6	1.175	0.565 ₀	-0.247 ₀	-0.686 ₀	-0.170 ₀	-0.685 ₇
0.7	1.235 ₈	0.646 ₀	-0.317 ₅	-0.720 ₃	-0.246 ₀	-0.828 ₈
0.8	1.304	0.722 ₀	-0.390 ₃	-0.732 ₁	-0.336 ₁	-0.960 ₀
0.9	1.380	0.791 ₀	-0.463 ₃	-0.724 ₀	-0.438 ₀	-1.074 ₈
1.0	1.462	0.855 ₀	-0.534 ₀	-0.698 ₈	-0.550 ₃	-1.167 ₅
1.1	1.551	0.913 ₃	-0.602 ₃	-0.658 ₈	-0.670 ₇	-1.235
1.2	1.644 ₈	0.964 ₀	-0.665 ₀	-0.606 ₅	-0.796 ₀	-1.275 ₀
1.3	1.743 ₈	1.011	-0.723 ₈	-0.545 ₃	-0.924 ₀	-1.287 ₈
1.4	1.846 ₈	1.051	-0.774 ₇	-0.477 ₇	-1.053	-1.272
1.5	1.953 ₈	1.086	-0.818 ₀	-0.406 ₅	-1.178	-1.230
1.6	2.063 ₈	1.116	-0.856 ₀	-0.334 ₀	-1.298	-1.164
1.7	2.176 ₈	1.141 ₅	-0.885 ₈	-0.262 ₄	-1.410 ₈	-1.078 ₈
1.8	2.292	1.163	-0.908 ₅	-0.193 ₃	-1.513	-0.977 ₈
1.9	2.409	1.181	-0.924 ₀	-0.127 ₀	-1.605 ₈	-0.863 ₀
2.0	2.528	1.196	-0.934 ₃	-0.067 ₄	-1.686	-0.741 ₈
2.1	2.648	1.208 ₅	-0.938 ₈	-0.012 ₄	-1.753 ₈	-0.616 ₈
2.2	2.769 ₈	1.218 ₈	-0.936 ₀	+0.036 ₀	-1.809	-0.491 ₀
2.3	2.892	1.226 ₈	-0.931 ₁	0.080 ₁	-1.852	-0.370
2.4	3.015	1.233	-0.921 ₁	0.117 ₈	-1.883	-0.254 ₈
2.5	3.138	1.238	-0.907 ₇	0.149 ₇	-1.903	-0.146 ₈
2.6	3.262	1.242	-0.891 ₃	0.176 ₇	-1.913	-0.041 ₁
2.7	3.386 ₈	1.245	-0.872 ₅	0.199 ₈	-1.913	+0.040 ₁
2.8	3.511	1.247	-0.851 ₈	0.217 ₈	-1.905	0.117 ₀
2.9	3.636	1.249	-0.829 ₁	0.232 ₅	-1.890	0.185 ₈
3.0	3.761	1.250	-0.805 ₈	0.244 ₈	-1.868 ₈	0.242 ₀
3.1	3.886	1.251	-0.780	0.253 ₀	-1.841 ₅	0.291 ₃
3.2	4.011	1.251 ₈	-0.754 ₅	0.261 ₈	-1.810 ₈	0.331 ₈
3.3	4.136	1.252	-0.728 ₁	0.266 ₀	-1.775 ₈	0.364 ₂
3.4	4.261 ₈	1.252 ₈	-0.701 ₈	0.271 ₈	-1.738	0.390 ₈
3.5	4.387	1.253	-0.673 ₀	0.274 ₄	-1.698	0.411 ₅
3.6	4.512	1.253	-0.646 ₃	0.276 ₈	-1.656	0.427 ₀
3.7			-0.618 ₈	0.278 ₈	-1.612	0.440 ₈
3.8			-0.590 ₀	0.279 ₀	-1.567 ₅	0.450 ₃
3.9			-0.562 ₀	0.280 ₈	-1.522	0.457 ₈
4.0			-0.534 ₅	0.281 ₅	-1.476	0.463 ₀
4.1			-0.506 ₃	0.281 ₀	-1.430	0.467 ₀
4.2			-0.478 ₁	0.282 ₃	-1.383	0.469 ₀
4.3			-0.449 ₀	0.282 ₅	-1.336	0.472 ₀
4.4			-0.421 ₀	0.282 ₈	-1.288 ₈	0.473 ₈
4.5			-0.393 ₃	0.282 ₇	-1.241	0.474 ₅
4.6			-0.365 ₁	0.282 ₈	-1.193 ₈	0.475 ₀
4.7					-1.146	0.475 ₇
4.8					-1.098	0.476 ₁
4.9					-1.051	0.476 ₃
5.0					-1.003	0.476 ₄

TABLE 5

ζ	g_0	g'_0	g_1	g'_1	g_2	g'_2	g_3	g'_3
0.0	0.588 ₉	0	-0.730 ₀	0	-0.758 ₈	0	-0.794 ₈	0
0.1	0.580 ₆	-0.153 ₉	-0.606 ₀	0.658 ₁	-0.709 ₈	-0.806 ₁	-0.770 ₁	0.483 ₉
0.2	0.550 ₀	-0.273 ₄	-0.602 ₉	1.180	-0.915 ₀	-1.471 ₃	-0.701 ₀	0.879 ₈
0.3	0.527 ₀	-0.361 ₃	-0.464 ₅	1.566	-1.088	-1.964	-0.508 ₇	1.142
0.4	0.487 ₇	-0.421 ₀	-0.204 ₀	1.824	-1.301 ₈	-2.267 ₈	-0.477 ₉	1.247
0.5	0.443 ₆	-0.455 ₈	-0.103 ₈	1.962	-1.535 ₃	-2.384	-0.354 ₆	1.194
0.6	0.397 ₈	-0.469 ₄	-0.095 ₀	1.996 ₃	-1.773	-2.332	-0.243 ₇	1.005
0.7	0.350 ₄	-0.465 ₃	0.202 ₆	1.943 ₈	-1.997	-2.140	-0.156 ₀	0.718 ₈
0.8	0.304 ₆	-0.447 ₄	0.481 ₄	1.822 ₃	-2.197	-1.846 ₃	-0.101 ₆	0.384 ₁
0.9	0.261 ₂	-0.419 ₀	0.655 ₃	1.653	-2.364 ₈	-1.493	-0.080 ₀	+0.051 ₄
1.0	0.221 ₉	-0.383 ₃	0.811 ₀	1.454	-2.495	-1.118	-0.080 ₀	-0.237 ₂
1.1	0.184 ₆	-0.344 ₁	0.945 ₉	1.242	-2.588 ₈	-0.755 ₆	-0.125 ₀	-0.452 ₃
1.2	0.152 ₃	-0.302 ₄	1.059 ₅	1.031	-2.647 ₈	-0.431 ₄	-0.177 ₄	-0.580 ₃
1.3	0.124 ₁	-0.261 ₅	1.152 ₃	0.830 ₃	-2.677	-0.161 ₉	-0.238 ₂	-0.622 ₃
1.4	0.099 ₈	-0.222 ₄	1.226	0.650 ₀	-2.682	-0.046 ₁	-0.299 ₄	-0.500 ₆
1.5	0.079 ₄	-0.186 ₇	1.283	0.492 ₂	-2.669 ₃	0.192 ₆	-0.354 ₅	-0.504 ₃
1.6	0.061 ₂	-0.154 ₁	1.325 ₅	0.359 ₈	-2.645	0.283 ₁	-0.399 ₂	-0.386 ₆
1.7	0.048 ₃	-0.125 ₄	1.356	0.251 ₁	-2.614 ₅	0.326 ₃	-0.431 ₄	-0.266 ₇
1.8	0.037 ₂	-0.100 ₆	1.376 ₃	0.165 ₈	-2.581	0.333 ₃	-0.450 ₈	-0.131 ₈
1.9	0.028 ₃	-0.079 ₇	1.390	0.101 ₀	-2.549	0.314 ₆	-0.458 ₄	-0.024 ₁
2.0	0.021 ₁	-0.062 ₂	1.397	0.053 ₈	-2.519	0.278 ₂	-0.456 ₄	+0.059 ₁
2.1	0.015 ₆	-0.048 ₀	1.401	+0.020 ₃	-2.493	0.234 ₀	-0.447 ₅	0.115 ₃
2.2	0.011 ₄	-0.036 ₃	1.402	-0.001 ₄	-2.472	0.187 ₇	-0.434 ₂	0.145 ₃
2.3	0.008 ₃	-0.027 ₃	1.401	-0.014 ₆	-2.456	0.143 ₈	-0.419 ₂	0.152 ₆
2.4	0.005 ₉	-0.020 ₄	1.399	-0.021 ₇	-2.443	0.105 ₁	-0.404 ₃	0.142 ₃
2.5	0.004 ₁	-0.014 ₉	1.397	-0.024 ₃	-2.434 ₃	0.073 ₁	-0.391 ₁	0.119 ₆
2.6	0.002 ₉	-0.010 ₃	1.394	-0.024 ₂	-2.428 ₃	0.048 ₂	-0.380 ₆	0.090 ₂
2.7	0.001 ₉	-0.007 ₇	1.392	-0.022 ₄	-2.425	0.030 ₀	-0.373 ₂	0.058 ₅
2.8	0.001 ₃	-0.005 ₄	1.390	-0.019 ₈	-2.422	0.017 ₆	-0.368 ₃	0.028 ₀
2.9	0.000 ₈	-0.003 ₈	1.388	-0.016 ₉	-2.421	0.010 ₁	-0.367 ₄	+0.001 ₃
3.0	0.000 ₃	-0.002 ₆	1.386 ₃	-0.014 ₀	-2.420	0.006 ₃	-0.368 ₄	-0.020 ₄
3.1	0.000 ₃	-0.001 ₆	1.385	-0.011 ₃	-2.419 ₃	0.005 ₁	-0.371 ₃	-0.036 ₈
3.2	0.000 ₂	-0.001 ₂	1.384	-0.009 ₃	-2.418 ₃	0.005 ₈	-0.375 ₆	-0.048 ₀
3.3	0.000 ₁	-0.000 ₈	1.383	-0.007 ₆	-2.418 ₃	0.007 ₁	-0.380 ₈	-0.054 ₆
3.4	0.000 ₁	-0.000 ₃	1.383	-0.006 ₂	-2.418	0.009 ₀	-0.386 ₄	-0.058 ₀
3.5	0.000 ₀	-0.000 ₄	1.382	-0.005 ₁	-2.417	0.011 ₁	-0.392 ₃	-0.058 ₆
3.6	0.000 ₀	-0.000 ₂	1.381 ₃	-0.004 ₄	-2.415 ₈	0.013 ₀	-0.398 ₁	-0.057 ₈
3.7	0.000 ₀	-0.000 ₁	1.381	-0.003 ₁	-2.414	0.014 ₇	-0.403 ₂	-0.055 ₂

TABLE 6

η	G_1/P	G'_1/P	G_2/P	G'_2/P	G_3/P	G'_3/P
0.05	1.269	-2.323 ₀	-1.799	10.835	-1.142	-7.924 ₅
0.10	1.156 ₅	-2.185	-1.319	8.542	-1.385	-2.555
0.15	1.051	-2.048	-0.932 ₄	6.999	-1.439	+0.138 ₂
0.2	0.951 ₇	-1.913 ₅	-0.613 ₉	5.789 ₈	-1.388 ₅	1.766 ₅
0.3	0.773 ₅	-1.653	-0.133 ₂	3.923	-1.115	3.445
0.4	0.620 ₇	-1.407	+0.186 ₃	2.530	-0.737 ₂	3.972
0.5	0.491 ₆	-1.180 ₅	0.384 ₁	1.476	-0.342 ₁	3.849
0.6	0.383 ₉	-0.975 ₂	0.490 ₄	0.601 ₁	+0.020 ₀	3.346
0.7	0.295 ₇	-0.793 ₂	0.529 ₇	+0.128 ₇	0.320 ₃	2.643
0.8	0.224 ₄	-0.634 ₉	0.522 ₂	-0.251 ₆	0.546 ₂	1.869
0.9	0.167 ₉	-0.500 ₀	0.484 ₂	-0.485 ₅	0.695 ₁	1.118
1.0	0.123 ₇	-0.387 ₂	0.428 ₈	-0.606 ₉	0.772 ₈	+0.454 ₁
1.1	0.089 ₈	-0.294 ₉	0.365 ₇	-0.644 ₂	0.790 ₀	-0.086 ₃
1.2	0.064 ₁	-0.220 ₅	0.301 ₉	-0.624 ₂	0.760 ₂	-0.487 ₂
1.3	0.045 ₁	-0.162 ₅	0.242 ₁	-0.567 ₉	0.697 ₁	-0.751 ₃
1.4	0.031 ₂	-0.117 ₅	0.189 ₀	-0.492 ₂	0.614 ₀	-0.893 ₁
1.5	0.021 ₂	-0.083 ₄	0.143 ₈	-0.409 ₉	0.521 ₈	-0.935 ₁
1.6	0.014 ₂	-0.058 ₂	0.106 ₉	-0.329 ₈	0.429 ₅	-0.902 ₄
1.7	0.009 ₄	-0.039 ₉	0.077 ₈	-0.257 ₂	0.343 ₀	-0.820 ₁
1.8	0.006 ₁	-0.026 ₅	0.055 ₁	-0.194 ₉	0.266 ₄	-0.710 ₀
2.0	0.002 ₄	-0.011 ₅	0.026 ₀	-0.103 ₃	0.148 ₅	-0.472 ₂
2.2	0.000 ₉	-0.004 ₆	0.011 ₃	-0.049 ₂	0.074 ₉	-0.274 ₆
2.4	0.000 ₃	-0.001 ₇	0.004 ₅	-0.021 ₆	0.034 ₄	-0.141 ₀
2.6	0.000 ₁	-0.000 ₆	0.001 ₆	-0.008 ₆	0.014 ₄	-0.065 ₉
2.8	0.000 ₀	-0.000 ₂	0.000 ₄	-0.003 ₁	0.005 ₅	-0.027 ₅
3.0		-0.000 ₁	0.000 ₂	-0.001 ₁	0.001 ₄	-0.010 ₅
3.2			0.000 ₁	-0.000 ₃	0.000 ₆	-0.003 ₆
3.4			0.000 ₀	-0.000 ₁	0.000 ₂	-0.001 ₁
3.6				0.000 ₀	0.000 ₀	-0.000 ₃
3.8						-0.000 ₁
4.0						0.000 ₀

AXISYMMETRIC SOLUTIONS OF THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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SUMMARY

The theory of steady axisymmetric solutions of the idealized hydromagnetic equations for a compressible gas is applied to an incompressible fluid. The aim of the paper is to investigate the solutions of the vorticity equation, which is a non-linear partial differential equation of the second order in ψ —the Stokes stream-function—and contains four arbitrary functions of ψ . The solutions of this equation in some particular choice of arbitrary functions are obtained, among which is the generalization of Hill's spherical vortex in the presence of a magnetic field.

Generally both the velocity and the magnetic field vectors become infinite on a certain surface defined by the singularity of the vorticity equation. Section 7 deals with this singularity.

1. Introduction

This paper is mainly inspired by T. V. Davies's paper (1) which was presented at 'The British Theoretical and Applied Mechanics Colloquium' held in Manchester, 6–9 April 1959. In that paper Davies deals with the case of a steady, compressible, inviscid fluid of infinite electrical conductivity, when both the magnetic and the velocity fields have an axial symmetry. Here that theory is applied to the case of an incompressible fluid.

As the reduction of magnetohydrodynamical equations is nearly the same as in (1), except that here the density ρ is constant, the details of this reduction have not been repeated here. The aim of this paper is to obtain the solutions of the vorticity equation (2.10) in different cases and to describe each motion corresponding to these solutions.

Generally both the velocity and the magnetic field vectors become infinite on a certain surface defined by the singularity of the vorticity equation. Section 7 deals with this singularity.

Chandrasekhar (3) has obtained the general equations, in terms of four arbitrary functions, as a particular case of his more general discussion.

But there no reference is made to the singularity, nor to any general relation showing the dependence of all arbitrary functions on each other, such as the vorticity equation (2.10).

2. The equations of motion

The equations of motion of magnetohydrodynamics are the union of Maxwell's electromagnetic equations and hydrodynamical equations, with the terms expressing the effect of interaction of fluid motion and magnetic field. In the case of a steady, incompressible, non-viscous fluid motion under the effect of a steady magnetic field these equations are

$$\mathbf{V} \times (\nabla \times \mathbf{V}) + \frac{1}{\rho} \mathbf{j} \times \mathbf{H} = \nabla \left(\frac{p}{\rho} + \frac{1}{2} q^2 + \Omega \right), \quad \nabla \cdot \mathbf{V} = 0, \quad (2.1)$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \quad \nabla \cdot \mathbf{j} = 0, \quad (2.2)$$

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0, \quad (2.3)$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{H}), \quad (2.4)$$

where \mathbf{V} is the velocity vector, \mathbf{H} the magnetic field vector, \mathbf{j} the current density vector, Ω the gravitational potential, p the pressure, ρ the constant density, \mathbf{E} the electric intensity vector, σ the electrical conductivity, and $q = |\mathbf{V}|$.

When the electrical conductivity is infinite (2.4) gives

$$\mathbf{E} = -\mathbf{V} \times \mathbf{H}; \quad (2.5)$$

hence (2.3) becomes $\nabla \times (\mathbf{V} \times \mathbf{H}) = 0. \quad (2.6)$

Eliminating \mathbf{j} between (2.1) and (2.2) we have

$$\mathbf{V} \times (\nabla \times \mathbf{V}) - \frac{1}{4\pi\rho} \mathbf{H} \times (\nabla \times \mathbf{H}) = \nabla \left(\frac{p}{\rho} + \frac{1}{2} q^2 + \Omega \right). \quad (2.7)$$

Equations (2.6) and (2.7) are the basic equations to be solved under the conditions that the vectors \mathbf{V} , \mathbf{H} , and \mathbf{j} are all solenoidal. In cylindrical polar coordinates r, θ, z , when \mathbf{V} and \mathbf{H} are azimuth-independent, it is possible to show that one can obtain a Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2} q^2 + \Omega - \frac{r H_2 f''}{4\pi\rho} = F(\psi), \quad (2.8)$$

and the vorticity equation

$$\begin{aligned} 4\pi\rho F' + \frac{1}{r^2} (g^2 - 4\pi\rho) \left(\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} \right) + \frac{1}{r^2} g g' (\psi_r^2 + \psi_z^2) \\ = \frac{1}{r} v K' + v H_2 g' - r H_2 f'', \end{aligned} \quad (2.9)$$

or

$$4\pi\rho r^2 F' + A \left(\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} \right) + gg'(\psi_r^2 + \psi_z^2) \\ = \frac{1}{A} \left\{ -K'(K - r^2 gf') + r^2 f''(gK - 4\pi\rho r^2 f') + \frac{g'}{A} (K - r^2 gf')(gK - 4\pi\rho r^2 f') \right\}, \quad (2.10)$$

where $A = g^2 - 4\pi\rho$, ψ is the Stokes stream-function introduced by $\nabla \cdot \mathbf{V} = 0$, f, g, K, F are arbitrary functions of ψ , F' the first derivative of F with respect to ψ , and ψ is an unknown function of r and z . Also u, v, w are the velocity components, H_1, H_2, H_3 the components of the magnetic field \mathbf{H} in the principal directions in cylindrical coordinates, and suffixes denote partial derivatives of ψ . See (1).

Now the vorticity equation (2.10) is a partial differential equation of the second order in ψ , containing four arbitrary functions of ψ , which are at our disposal. By choosing these functions suitably we can solve the equation and express ψ in terms of r and z . Then

$$\left. \begin{aligned} u &= -\frac{1}{r} \psi_z, & v &= -\frac{1}{r} \frac{K - r^2 gf'}{g^2 - 4\pi\rho}, & w &= \frac{1}{r} \psi_r \\ H_1 &= -\frac{1}{r} g \psi_z, & H_2 &= -\frac{1}{r} \frac{gK - 4\pi\rho r^2 f'}{g^2 - 4\pi\rho}, & H_3 &= \frac{1}{r} g \psi_r \\ j_1 &= \frac{1}{4\pi\rho} \frac{\partial}{\partial z} (r H_2), & j_2 &= \frac{1}{4\pi} (H_{1z} - H_{3r}), & j_3 &= \frac{1}{4\pi\rho} \frac{\partial}{\partial r} (r H_2) \end{aligned} \right\} \quad (2.11)$$

follows. Bernoulli's equation (2.8) determines the pressure.

It is clear that $\mathbf{H} = g\mathbf{V}$, if and only if $H_2 = gv$, i.e. if $f' = 0$. Then the vectors \mathbf{V} and \mathbf{H} are everywhere parallel. The coefficient of the highest derivative in the vorticity equation is $A = g^2 - 4\pi\rho$. On the singular surface $g^2 - 4\pi\rho = 0$, which is generally a surface of revolution about the axis of z , both the velocity and magnetic fields tend to infinity. See section 7.

3. Particular cases

In this and the following sections we shall try to find particular solutions of the vorticity equation (2.10), which is a non-linear partial differential equation of the second order in ψ . The equation contains four arbitrary functions of ψ , namely, $f(\psi)$, $g(\psi)$, $K(\psi)$, $F(\psi)$, and ψ is an unknown function of r and z .

Beginning with the simplest cases, we shall show that, choosing all or some of the arbitrary functions suitably the vorticity equation can be solved to determine ψ . After determining the velocity field we shall give

a general picture of the motion and find the magnetic and the current density fields associated with the solution.

3.1. Let $K' = g' = f' = F' = 0$

Let $K = K_0$, $g = g_0$, $f' = f'_0$, and $F = F_0$, where K_0 , g_0 , f'_0 , F_0 are arbitrary constants. The equation (2.11) shows that v and H_z are functions of r only. Now the vorticity equation (2.10) reduces to

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = 0, \quad (3.1)$$

and we shall assume that $g^2 - 4\pi\rho$ is not identically zero.

(i) One well-known solution of this equation is (2)

$$\psi = \frac{1}{2}Ur^2 \left[1 - \left(\frac{a^2}{r^2 + z^2} \right)^{\frac{3}{2}} \right], \quad (3.2)$$

where a , U are arbitrary constants; from (3.2) we can obtain the components u and w of the velocity outside the sphere $r^2 + z^2 = a^2$, which are finite at infinity. The component v of the velocity is, by (2.11),

$$v = -\frac{1}{r} \frac{K_0 - r^2 g_0 f'_0}{g_0^2 - 4\pi\rho}.$$

If we assume that $f'_0 = 0$, v becomes zero at infinity. Hence we have

$$\left. \begin{aligned} u &= -\frac{3}{2} \frac{Ua^3 rz}{(r^2 + z^2)^{\frac{5}{2}}} \\ v &= -\frac{K_0}{g_0^2 - 4\pi\rho} \frac{1}{r} \\ w &= U \left[1 - \frac{a^3}{(r^2 + z^2)^{\frac{5}{2}}} + \frac{3}{2} \frac{a^3 r^2}{(r^2 + z^2)^{\frac{7}{2}}} \right] \end{aligned} \right\} \quad (3.3)$$

The velocity is infinite at the origin, but if the origin is enclosed in any sphere of radius a , the velocity outside of the sphere is finite everywhere except on the axis. At infinity $u = w = 0$, and the motion which is irrotational everywhere reduces to a parallel flow in the direction of z , the magnitude of the velocity being U .

Since $\psi = 0$ on the surface of the sphere, this surface is covered by stream lines. They are helices traced on the sphere. The stream lines outside the sphere are again helices, but as r increases both v and u diminish and the motion sufficiently far from the z -axis becomes parallel to the axis.

Since $f'_0 = 0$, (2.11) shows that $\mathbf{H} = g_0 \mathbf{V}$, that is the magnetic field vector is everywhere parallel to the velocity vector, and is a constant multiple of it in magnitude. The magnetic lines of force are the same as the stream lines whilst the current vector \mathbf{j} is identically zero.

(ii) Suppose that ψ is of the form

$$\psi = R(r) \cdot Z(z),$$

then (3.1) becomes

$$\frac{R'' - R'/r}{R} + \frac{Z''}{Z} = 0.$$

The first term is a function of r and the second term is a function of z only; hence each term is a constant, m say, so that

$$\left. \begin{aligned} Z'' - mZ &= 0 \\ R'' - R'/r + mR &= 0 \end{aligned} \right\} \quad (3.4)$$

and

where m is an arbitrary constant which may be positive, negative, or zero.

(a) If $m = n^2$, we have

$$Z = Ae^{nz} + Be^{-nz},$$

and the equation in R transforms to a Bessel's equation by writing $R = rP$, where P is a function of r , namely,

$$r^2 P'' + rP' + (n^2 r^2 - 1)P = 0.$$

Hence

$$P = R/r = J_1(nr),$$

and

$$R = rJ_1(nr).$$

The second solution of Bessel's equation is omitted since the velocity is finite on the axis. Hence

$$\left. \begin{aligned} \psi &= rJ_1(nr)(Ae^{nz} + Be^{-nz}) \\ u &= -nJ_1(nr)(Ae^{nz} - Be^{-nz}) \\ v &= \frac{g_0 f'_0}{g_0^2 - 4\pi\rho} r = \varpi r \quad \text{if } K_0 = 0 \\ w &= nJ_0(nr)(Ae^{nz} + Be^{-nz}) \end{aligned} \right\} \quad (3.5)$$

n being chosen positive. The velocity is along the axis and is finite on the axis, except at infinity; the arbitrary constants A , B , n , and ϖ are determined by the boundary conditions. The motion is generally helical.

(b) If $m = -n^2$, we find

$$Z = A \cos nz + B \sin nz,$$

$$R = rI_1(nr),$$

and

$$\psi = rI_1(nr)(A \cos nz + B \sin nz).$$

Hence

$$\left. \begin{aligned} u &= nI_1(nr)(A \sin nz - B \cos nz) \\ v &= \frac{g_0 f'_0}{g_0^2 - 4\pi\rho} r = \varpi r \quad \text{if } K_0 = 0 \\ w &= nI_0(nr)(A \cos nz + B \sin nz) \end{aligned} \right\} \quad (3.6)$$

where I_1 and I_0 are Bessel's functions of imaginary argument.

(c) If $m = 0$, we find

$$\text{so that } \left. \begin{aligned} \psi &= (Ar^2 + B)(Cz + D) \\ u &= -\frac{C}{r}(Ar^2 + B) \\ v &= \varpi r \quad \text{if } K_0 = 0 \\ w &= 2A(Cz + D) \end{aligned} \right\}, \quad (3.7)$$

where A, B, C, D are arbitrary constants and

$$\varpi = g_0 f'_0 / (g_0^2 - 4\pi\rho) = \text{constant},$$

which is also arbitrary. If $B = 0$ we find

$$\left. \begin{aligned} u &= -ACr \\ v &= \varpi r \\ w &= 2A(Cz + D) \end{aligned} \right\}. \quad (3.8)$$

When $r = 0$, $u = v = 0$, and w is only a function of z ; hence the velocity on the axis is along it and a function of z . When $z = -D/C$, $w = 0$, and the stream lines on this plane are equiangular spirals, w is positive on one side of the plane and negative on the other. The stream lines are given by the equation

$$\frac{dr}{-ACr} = \frac{r d\theta}{\varpi r} = \frac{dz}{2A(Cz + D)}.$$

The first two ratios give $r = ae^{-(AC/\varpi)\theta}$, (3.9)

the first and third give $r^2(Cz + D) = b$, (3.10)

where a and b are arbitrary constants of integration; (3.9) represents a family of cylinders with their generators parallel to the axis of z , and whose projections on the planes $z = \text{constant}$ are equiangular spirals given by (3.9) itself. The constant slope of these spirals to the radial vectors through the axis is $-\varpi/AC$. (3.10) represents a family of surfaces of revolution about the axis. The actual stream lines are the intersections of these two families of surfaces, the velocity being infinite at infinity. The magnetic field and electric current components are

$$\begin{aligned} H_1 &= -ACg_0 r, & H_2 &= \frac{4\pi\rho}{g_0} \varpi r, & H_3 &= 2Ag_0(Cz + D), \\ j_1 &= 0, & j_2 &= 0, & j_3 &= \frac{2\rho}{g_0} \varpi, \end{aligned}$$

where $\varpi = g_0 f'_0 / (g_0^2 - 4\pi\rho)$.

3.2. Let $K' = g' = f'' = 0$ and $4\pi\rho F' = (4\pi\rho - g_0^2)C$, where C is an arbitrary constant

The equation (2.10) becomes

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = Cr^2. \quad (3.11)$$

One known solution of this linear equation is (2)

$$\psi = \psi_1 = -\frac{1}{10}Cr^2(a^2 - r^2 - z^2),$$

where a is an arbitrary constant. Since $\psi = 0$ on the surface of the sphere $r^2 + z^2 = a^2$, the surface of the sphere is covered with stream lines: no flow passes the surface. But u and w become infinitely large at infinity, so that the solution is valid inside the sphere only and we take the steady flow given by (3.3) outside.

Now outside the region ψ is given by

$$\psi_2 = \frac{1}{2}Ur^2 \left[1 - \left(\frac{a^2}{r^2 + z^2} \right)^{\frac{3}{2}} \right], \quad (3.2)$$

where $r^2 + z^2 \geq a^2$ and U is the constant velocity parallel to Oz at infinity. This is a solution of the equation (3.11) when $C = 0$. The two expressions for ψ show that the normal velocity is zero on both sides of the surface of the sphere, and hence is continuous. The continuity of the tangential velocities require that $C = 15U/2a^2$. Hence, inside the sphere,

$$\psi_1 = -\frac{3U}{4a^2}r^2(a^2 - r^2 - z^2), \quad (3.12)$$

where $r^2 + z^2 \leq a^2$.

In this case

$$v = -\frac{1}{r} \frac{K_0 - r^2 g_0 f'_0}{g_0^2 - 4\pi\rho}.$$

If $K_0 = f'_0 = 0$, then $v = 0$, and (3.12) gives the Hill's spherical vortex (2) inside the sphere $r^2 + z^2 = a^2$. And (3.2) represents an irrotational steady flow outside the sphere, which is finite and equal to U parallel to Oz at infinity. The motion is meridional everywhere. The stream surfaces are given by $\psi = \text{constant}$. Inside the sphere stream lines form closed curves in meridional planes around the points $z = 0$, $r = a/\sqrt{2} = 0.707a$.

Since $f'_0 = 0$, we have $\mathbf{H} = g_0\mathbf{V}$; thus the magnetic lines of force are coincident with stream lines. Current density lines are circles around the axis in planes $z = \text{constant}$, whose centres are on the axis. Its magnitude is $-15Ug_0r/8\pi a^2$ inside, and zero outside the sphere.

If $K_0 = 0$, $f'_0 \neq 0$, then $v = \{g_0 f'_0 / (g_0^2 - 4\pi\rho)\}r = \varpi r$, say. This adds a rotation, with constant angular velocity ϖ , to the motion described above. The stream lines become helices traced on coaxial surfaces of revolution $\psi_1 = \text{constant}$ inside and $\psi_2 = \text{constant}$ outside.

Magnetic lines of force are helices similar to the stream lines, but they are not coincident.

If $K_0 \neq 0$, $f'_0 = 0$, then $v = -(1/r)K_0/(g_0^2 - 4\pi\rho)$. This adds a rotation to the meridional motion defined by ψ_1 and ψ_2 , which is infinite on the axis and tends to zero when $r \rightarrow \infty$. The stream lines are helices as before, the magnetic lines are the same as the stream lines since $\mathbf{H} = g_0 \mathbf{V}$.

If one uses

$$v_1 = \frac{g_0 f'_0}{g_0^2 - 4\pi\rho} r \quad \text{inside the sphere,}$$

$$v_2 = \frac{-1}{r} \frac{K_0}{g_0^2 - 4\pi\rho} \quad \text{outside the sphere,}$$

so that v is finite everywhere, the v -components of the velocity and pressure on both sides of the surface of the sphere cannot be made equal everywhere, hence an elastic spherical membrane cannot remain spherical during the motion.

4. Solution of the vorticity equation when $\psi_r = 0$

Equations (2.11) show that in this case $w = H_3 = 0$. Hence the motion takes place in planes perpendicular to the axis of z and the magnetic field lines are in the same planes as the stream lines.

When $\psi_r = 0$ the equation (2.10) can be written

$$8\pi\rho r^2 F' + [(g^2 - 4\pi\rho)\psi_z^2]' + \left(\frac{K^2}{A}\right)' + 4\pi\rho r^4 \left(\frac{f'^2}{A}\right)' = r^2 \left[\frac{2g(Kf')'}{A} - \frac{2Kg'f'(g^2 + 4\pi\rho)}{A^2} \right]. \quad (4.1)$$

Since F , f , g , and K are functions of ψ , which is a function of z only, (4.1) is satisfied only if the coefficients of powers of r are the same on both sides. Thus we find

$$f'^2 = B(g^2 - 4\pi\rho), \quad (4.2)$$

$$(g^2 - 4\pi\rho)\psi_z^2 + \frac{K^2}{g^2 - 4\pi\rho} = C, \quad (4.3)$$

$$4\pi\rho(g^2 - 4\pi\rho)F' = g(Kf')' - Kf'g' \frac{g^2 + 4\pi\rho}{g^2 - 4\pi\rho}, \quad (4.4)$$

where B and C are arbitrary constants of integration and $C \neq 0$. These three equations must be satisfied simultaneously. If $F' \equiv 0$, (4.4) is satisfied by

$$(a) \quad f' = 0, \quad (b) \quad g = 0, \quad (c) \quad K = 0, \quad (4.5)$$

and lastly if f' , g , $K \neq 0$ by

$$(d) \quad f'gK = E(g^2 - 4\pi\rho), \quad (4.6)$$

where E is an arbitrary constant.

In case (a), $F' \equiv 0$, $B = 0$ (i.e. $f' = 0$).

The equation (4.2) gives $f' = 0$ and (4.4) is satisfied. (4.3) gives

$$\psi_z = \frac{\mp [C(g^2 - 4\pi\rho) - K^2]^{\frac{1}{2}}}{g^2 - 4\pi\rho}. \quad (4.7)$$

By choosing $g(\psi)$ and $K(\psi)$ we can determine $\psi(z)$, and hence u , in terms of z . But since

$$\left. \begin{aligned} u &= \pm \frac{1}{r} \frac{[C(g^2 - 4\pi\rho) - K^2]^{\frac{1}{2}}}{g^2 - 4\pi\rho} \\ v &= -\frac{1}{r} \frac{K}{g^2 - 4\pi\rho}, \quad w = 0 \end{aligned} \right\} \quad (4.8)$$

the stream lines are given by $dr/u = r d\theta/v = dz/w$. Hence

$$z = c_1, \quad r = c_2 e^{k\theta}, \quad (4.9)$$

where $k = \mp [C(g^2 - 4\pi\rho) - K^2]^{\frac{1}{2}}/K = \text{a function of } z \text{ only}$. Thus stream lines are spirals in the planes $z = \text{constant}$, which make an angle α with the radius vector at the point considered such that

$$\tan \alpha = \frac{r d\theta}{dr} = \frac{\mp K}{[C(g^2 - 4\pi\rho) - K^2]^{\frac{1}{2}}} = \frac{1}{k}$$

is a function of z only. Thus this angle is constant in each plane perpendicular to the axis, but changes with z . The motion is real only if

$$C(g^2 - 4\pi\rho) - K^2 \geq 0,$$

which defines a region or regions of validity. Since $f' = 0$,

$$\mathbf{H} = g\mathbf{V}, \quad (4.10)$$

i.e. the magnetic and the velocity field vectors are parallel; hence the magnetic field lines are coincident with the stream lines. The velocity is infinite on the axis and on the planes $g^2 - 4\pi\rho = 0$.

The cases (b), (c), and (d) give similar motions.

5. Solution of the vorticity equation when $\psi_z = 0$

The equations (2.11) show that $u = H_1 = 0$. Hence the stream lines, and the magnetic field lines lie on circular cylinders coaxial with the axis of z . They are generally helices, whose inclination to the generators of the cylinders is a function of r , and so constant on each cylinder.

In this case the vorticity equation (2.10) can be written as

$$\begin{aligned} 8\pi\rho r^2 F' + r^2[(g^2 - 4\pi\rho)w^2]' + \left(\frac{K^2}{A}\right)' + 4\pi\rho r^4 \left(\frac{f'^2}{A}\right)' \\ = r^2 \left[\frac{2g(Kf)'}{A} - \frac{2Kf'g'(g^2 + 4\pi\rho)}{A^2} \right], \end{aligned} \quad (5.1)$$

where $A = g^2 - 4\pi\rho$, supposed not identically zero, and $w = (1/r)\psi_1$ is the velocity component in the direction of z . Dashes refer to differentiation with respect to ψ , and ψ is a function of r only. Some particular cases are discussed in the next section.

5.1. $K = 0$

In this case (2.11) shows that

$$v = (g/4\pi\rho)H_2,$$

i.e. v and H_2 are proportional, the coefficient of proportionality being a function of r . The equation (5.1) becomes

$$8\pi\rho F' + [(g^2 - 4\pi\rho)w^2]' + 4\pi\rho r^2 \left(\frac{f'^2}{A} \right)' = 0. \quad (5.2)$$

Now let us change all the derivatives with respect to ψ , to derivatives with respect to r by using the operator

$$\psi_r \frac{d}{d\psi} = \frac{d}{dr}.$$

Hence multiplying by ψ_r throughout and using the operator, we get

$$8\pi\rho F_r + [w^2(g^2 - 4\pi\rho)]_r + 4\pi\rho r^2 \left(\frac{f_r^2/r^2}{w^2(g^2 - 4\pi\rho)} \right)_r = 0. \quad (5.3)$$

Now if we write

$$w^2(g^2 - 4\pi\rho) = y, \quad (5.4)$$

(5.3) becomes

$$y_r(y^2 - 4\pi\rho f_r^2) + 4\pi\rho y \left[(f_r^2)_r - \frac{2}{r}f_r^2 + 2yF_r \right] = 0. \quad (5.5)$$

The formulae for v and H_2 are

$$\left. \begin{aligned} v &= \frac{rgf'_r}{g^2 - 4\pi\rho} = \frac{rgf_r}{(g^2 - 4\pi\rho)\psi_r} = \frac{gf_r}{(g^2 - 4\pi\rho)w} \\ H_2 &= \frac{4\pi\rho rf'_r}{g^2 - 4\pi\rho} = \frac{4\pi\rho f_r}{(g^2 - 4\pi\rho)w} = \frac{4\pi\rho}{g}v \end{aligned} \right\}. \quad (5.6)$$

Now equation (5.5) can be satisfied by

$$(f_r^2)_r - \frac{2}{r}f_r^2 + 2yF_r = 0, \quad (5.7)$$

and

$$\begin{cases} y_r = 0 \\ \text{or } y^2 - 4\pi\rho f_r^2 = 0; \end{cases} \quad (5.8)$$

$$(5.7) \text{ gives } F, \text{ and from (5.8) we obtain} \quad (5.9)$$

$$\left. \begin{aligned} w &= \frac{\mp a}{[a(g^2 - 4\pi\rho)]^{\frac{1}{2}}} \\ v &= \frac{\mp gf_r}{[a(g^2 - 4\pi\rho)]^{\frac{1}{2}}} \end{aligned} \right\}, \quad (5.10)$$

where g and f_r are arbitrary, and a is an arbitrary constant. The stream lines are

$$r = c_1, \quad z = \frac{ar}{gf_r} \theta + c_2, \quad (5.11)$$

that is, helices traced on coaxial cylinders $r = c_1$, whose inclination to the generators $v/w = gf_r/a$ is a function of r .

The magnetic field and electric current components are

$$\left. \begin{aligned} H_1 &= 0, & H_2 &= \frac{\mp 4\pi\rho f_r}{[a(g^2 - 4\pi\rho)]^{\frac{1}{2}}}, & H_3 &= \frac{\mp ga}{[a(g^2 - 4\pi\rho)]^{\frac{1}{2}}} \\ j_1 &= 0, & j_2 &= -\frac{1}{4\pi} H_{3r}, & j_3 &= \frac{1}{4\pi r} (rH_2)_r \end{aligned} \right\} \quad (5.12)$$

If we use (5.9) instead of (5.8) we get the set of equations

$$(f_r^2)_r - \frac{2}{r} f_r^2 + 2yF_r = 0, \quad y^2 - 4\pi\rho f_r^2 = 0.$$

The first defines F , the second gives

$$y = w^2(g^2 - 4\pi\rho) = \mp \sqrt{(4\pi\rho)} f_r;$$

hence

$$\left. \begin{aligned} w &= \mp (4\pi\rho)^{\frac{1}{2}} \left(\frac{\mp f_r}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}} \\ v &= \mp \frac{g}{(4\pi\rho)^{\frac{1}{2}}} \left(\frac{\mp f_r}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}} \end{aligned} \right\} \quad (5.13)$$

a solution identical with (5.10) when $\sqrt{(4\pi\rho)} f_r = a$. The stream lines

$$r = c_1, \quad z = \frac{\sqrt{(4\pi\rho)}}{g} r\theta + c_2 \quad (5.14)$$

are again helices on the cylinder $r = c_1$. If for a certain value of r (say r_0), $f_r = 0$, and if $[(g^2 - 4\pi\rho)/g^2]_{r=r_0} \neq 0$, then both v and w are zero on the cylinder $r = r_0$. If, for instance,

$$\frac{\mp f_r}{g^2 - 4\pi\rho} = \sqrt{(4\pi\rho)} a^2 (r - r_0)^2,$$

and if we take $+$ signs, we have

$$\left. \begin{aligned} w &= \sqrt{(4\pi\rho)} a (r - r_0) \\ v &= ga (r - r_0) \end{aligned} \right\} \quad (5.15)$$

Now if $g(r_0)$ is finite, both v and w become zero when $r = r_0$; i.e. the velocity is zero on the cylinder $r = r_0$, and has opposite signs inside and outside this cylinder.

The stream lines are given by (5.14) everywhere. If we assume $g(r) = r$, and $a < 0$, the velocity on the axis is

$$w = -\sqrt{(4\pi\rho)} ar_0, \quad v = 0,$$

i.e. the fluid particles on the axis have only a translational motion along the axis, which is constant and positive. As r increases, w decreases and tends to zero as $r \rightarrow r_0$. On the other hand, since $g(r) = r$, and $a < 0$, v increases first, takes a maximum value, and then decreases again to zero as $r \rightarrow r_0$.

Outside the cylinder, $r > r_0$, both v and w change sign to become negative. The velocity is zero on the cylinder $r = r_0$, but for $r > r_0$, v and w become negative and increase indefinitely as $r \rightarrow \infty$.

The components of the magnetic field and electric current are

$$\left. \begin{aligned} H_1 &= 0, & H_2 &= 4\pi\rho a(r-r_0), & H_3 &= \sqrt{(4\pi\rho)ar(r-r_0)} \\ j_1 &= 0, & j_2 &= -\left(\frac{\rho}{4\pi}\right)^{\frac{1}{2}}ar, & j_3 &= \rho a \frac{2r-r_0}{r} \end{aligned} \right\}. \quad (5.16)$$

Hence the magnetic field vector \mathbf{H} is also zero when $r = r_0$. The current vector is infinite when $r = 0$, it is constant when $r = r_0$, and j_3 changes sign when $r = \frac{1}{2}r_0$.

Again, if we assume that

$$\frac{\mp f_r}{g^2 - 4\pi\rho} = \sqrt{(4\pi\rho)a^2 \sin^2 r},$$

and if $g(0)$ is finite, by taking the $+$ signs only we have (see (5.13))

$$\left. \begin{aligned} w &= \sqrt{(4\pi\rho)a \sin r} \\ v &= ga \sin r \end{aligned} \right\}. \quad (5.17)$$

The velocity is zero when $r = 0, \pi, 2\pi, 3\pi, \dots$. Hence the cylinders $r = k\pi$ (k integral) can be taken as boundaries and the motion between them is then known.

The stream lines are everywhere the helices given by (5.14), on cylinders coaxial with the axis of z . Their inclinations to the generators of the cylinders are constant and equal to $v/w = g/\sqrt{(4\pi\rho)}$. The total velocity q is given by

$$q = (v^2 + w^2)^{\frac{1}{2}} = (g^2 + 4\pi\rho)^{\frac{1}{2}}a \sin r.$$

Therefore the velocity is zero when $r = 0, \pi, 2\pi, \dots$ and a maximum when $r = \pi/2, 3\pi/2, 5\pi/2, \dots$. The maximum value of the velocity is $(g^2 + 4\pi\rho)^{\frac{1}{2}}a$, which is a function of r provided that g is also. The velocity changes its sign as each bounding cylinder (the cylinder on which the velocity is zero) is passed. The magnetic field and current components are

$$\left. \begin{aligned} H_1 &= 0, & H_2 &= 4\pi\rho a \sin r, & H_3 &= \sqrt{(4\pi\rho)ga \sin r} \\ j_1 &= 0, & j_2 &= -\left(\frac{\rho}{4\pi}\right)^{\frac{1}{2}}ga \cos r, & j_3 &= \rho a \left(\frac{\sin r}{r} + \cos r\right) \end{aligned} \right\}. \quad (5.18)$$

The magnetic field vector \mathbf{H} is zero on the bounding cylinders formed of the stream lines, and a maximum on the intermediate cylinders.

5.2. $g = 0$

In this case $g^2 - 4\pi\rho$ is always different from zero and there is no singularity. Then (2.11) shows that $H_1 = H_3 = 0$, $H_2 = -rf'$. Therefore magnetic field lines are circles about the axis of z , instead of helices.

The equation (5.1) becomes

$$8\pi\rho r^2 F' - 4\pi\rho r^2 (w^2)' = \frac{1}{4\pi\rho} [(K^2)' + 4\pi\rho r^4 (f'^2)'],$$

and after transforming all the derivatives from ψ to r as in section 5.1 this can be written as

$$(4\pi\rho r)^2 (2F - w^2)_r = (K^2)_r + 4\pi\rho r^4 \left(\frac{f_r^2}{r^2 w^2} \right)_r. \quad (5.19)$$

$$\text{If } \frac{f_r}{rw} = a = \text{constant}, \quad (5.20)$$

then (5.19) becomes

$$(4\pi\rho r)^2 (2F - w^2)_r = (K^2)_r, \quad (5.21)$$

and (5.20) gives

$$\left. \begin{aligned} w &= \frac{1}{ar} f_r \\ v &= \frac{K}{4\pi\rho r} \end{aligned} \right\}. \quad (5.22)$$

whence, from (2.11),

f_r and K are arbitrary functions of r and (5.21) defines F .

If for example we choose F and K so that $F = bf$ and $K = 4\pi\rho f$, the relation (5.21) becomes

$$r^2 \left[2bf_r - \frac{2f_r f_{rr}}{a^2 r^2} + \frac{2f_r^2}{a^2 r^3} \right] = 2ff_r.$$

Hence either $f_r = 0$, or

$$f_{rr} - \frac{1}{r} f_r + a^2 f = a^2 b r^2. \quad (5.23)$$

If $f_r = 0$, we obtain $w = 0$, $v = (1/r)f_0$, which represents a pure rotation.

If $f_r \neq 0$, we have to solve (5.23). If we put $f = rP$, this becomes

$$r^2 P_{rr} + rP_r + (a^2 r^2 - 1)P = a^2 b r^3,$$

which is Bessel's equation of order one. Thus

$$P = \frac{1}{r} f = cJ_1(ar) + br,$$

where a, b, c are all constants and

$$\left. \begin{aligned} w &= cJ_0(ar) + \frac{2b}{a} \\ v &= cJ_1(ar) + br \end{aligned} \right\}. \quad (5.24)$$

[The second solution of Bessel's equation is rejected since w is finite on the axis.] The magnetic field and current components are

$$\left. \begin{aligned} H_1 = 0, \quad H_2 = -\frac{f_r}{w} = -ar, \quad H_3 = 0 \\ j_1 = 0, \quad j_2 = 0, \quad j_3 = -\frac{a}{2\pi} \end{aligned} \right\} \quad (5.25)$$

Both the magnetic field and current field are very simple. Magnetic field lines are circles with their centres on the axis in planes perpendicular to the axis. The current vector is constant and parallel to the axis. The solution (5.24) can be obtained directly from (5.21) by using $F' = b$, $f'' = 0$, $K = 4\pi\rho a\psi$.

5.3. $f' = 0$

The equation (2.10) shows that $H_2 = gv$; and since $H_1 = gu$, $H_3 = gw$, the magnetic field vector is parallel to the velocity vector everywhere. Therefore the stream lines and magnetic field lines are coincident.

Equation (5.1) becomes

$$8\pi\rho r^2 F' + r^2 [w^2(g^2 - 4\pi\rho)]' + \left(\frac{K^2}{g^2 - 4\pi\rho}\right)' = 0, \quad (5.26)$$

or if we transform the derivatives from ψ to r by $\psi_r \frac{d}{d\psi} = \frac{d}{dr}$, we get

$$8\pi\rho F_r + [w^2(g^2 - 4\pi\rho)]_r + \frac{1}{r^2} \left(\frac{K^2}{g^2 - 4\pi\rho}\right)_r = 0, \quad (5.27)$$

where $w = (1/r)\psi_r$ is the velocity component in the direction of z .

$$\left. \begin{aligned} \text{Also} \quad v &= -\frac{1}{r} \left(\frac{K}{g^2 - 4\pi\rho}\right) \\ \text{and} \quad H_1 = 0, \quad H_2 &= -\frac{1}{r} \frac{gK}{g^2 - 4\pi\rho}, \quad H_3 = gw, \\ j_1 = 0, \quad j_2 &= -\frac{1}{4\pi} (gw)_r, \quad j_3 &= -\frac{1}{4\pi r} \left(\frac{gK}{g^2 - 4\pi\rho}\right)_r \end{aligned} \right\} \quad (5.28)$$

Hence, as was stated at the end of section 2,

$$\mathbf{H} = g\mathbf{V}, \quad (5.29)$$

where g is a function of r .

The equation (5.27) is a relation between four arbitrary functions F , g , K , and w . It gives w in terms of the other three functions in the most general way, namely,

$$w^2(g^2 - 4\pi\rho) = C - 8\pi\rho F - \int \left(\frac{K^2}{g^2 - 4\pi\rho}\right)_r \frac{dr}{r^2}, \quad (5.30)$$

where C is a constant of integration, F , K , and g are arbitrary functions of r .

Let

$$\frac{K^2}{g^2 - 4\pi\rho} = ar^m,$$

where a and m are arbitrary constants. Then (5.30) gives

$$w = \mp \left(\frac{C - 8\pi\rho F - mar^{m-2}/(m-2)}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}} \quad (5.31)$$

and

$$v = \mp r^{(m-2)/2} \left(\frac{a}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}}$$

In this solution F and g are arbitrary functions of r and a , m , C are arbitrary constants. The stream lines are again helices traced on coaxial circular cylinders $r = c_1$ about the axis. Their inclination to the generators and their pitch are generally functions of r .

If, for instance, $m = 4$, $F = 0$, (5.31) becomes

$$w = \mp \left(\frac{C - 2ar^2}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}}, \quad v = \mp r \left(\frac{a}{g^2 - 4\pi\rho} \right)^{\frac{1}{2}},$$

C/a must be positive. Generally this defines a motion inside and on the surface of a certain cylinder $r \leq (C/2a)^{\frac{1}{2}}$. Also

$$\left. \begin{aligned} H_1 &= 0, & H_2 &= gv, & H_3 &= gw \\ j_1 &= 0, & j_2 &= -\frac{1}{4\pi}(gw)_r, & j_3 &= \frac{1}{4\pi r}(rgv)_r \end{aligned} \right\} \quad (5.32)$$

6. Motions in meridional planes

These are obtained by assuming $v = 0$. But since

$$v = -\frac{1}{r} \frac{K - r^2 gf'}{g^2 - 4\pi\rho},$$

we have $K - r^2 gf' \equiv 0$. Hence either

- (i) $K = g = 0$, or
- (ii) $K = f' = 0$.

In the first case the equation (2.10) becomes

$$4\pi\rho \left(\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} \right) + r^4 f' f'' = 4\pi\rho r^2 F', \quad (6.1)$$

and in the second it becomes

$$(g^2 - 4\pi\rho) \left(\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} \right) + gg'(\psi_r^2 + \psi_z^2) = -4\pi\rho r^2 F'. \quad (6.2)$$

Solutions of (6.1)

When $f'' = F' = 0$, (6.1) reduces to (3.1). Hence all the solutions of (3.1) obtained in section 3, with the additional condition $K = g = 0$, are also solutions of (6.1). For example (3.5), (3.6), and (3.8) when $g_0 = 0$.

When $f'' = 0$, $F' = C$, (6.1) reduces to (3.11). Hence (3.12) is a solution of (6.1) when $g_0 = 0$.

Also all the solutions obtained in sections 4 and 5 represent meridional motions when $v = 0$.

Solutions of (6.2)

Let $\psi = R(r) \cdot Z(z)$, $g^2 - 4\pi\rho = 2\psi$, i.e. $gg' = 1$, and $F' = 0$. Then (6.2) becomes

$$2RZ\left(R'Z - \frac{1}{r}R'Z + RZ''\right) + (R'Z)^2 + (RZ')^2 = 0.$$

Dividing by R^2Z^2 we find

$$\left(\frac{2Z''}{Z} + \frac{Z'^2}{Z^2}\right) + \left(\frac{2R''}{R} - \frac{2}{r}\frac{R'}{R} + \frac{R'^2}{R^2}\right) = 0.$$

The first group of terms is a function of z , and the second is a function of r only. Hence each must be a constant, m say, whence

$$\left. \begin{aligned} 2ZZ'' + Z'^2 - mZ^2 &= 0 \\ 2RR'' - \frac{2}{r}RR' + R'^2 + mR^2 &= 0 \end{aligned} \right\}. \quad (6.3)$$

The solutions of these equations, when $m = 0$, are easily found to be

$$Z = (Cz + D)^{\frac{1}{2}},$$

$$R = (Ar^2 + B)^{\frac{1}{2}};$$

hence

$$\psi = (Ar^2 + B)^{\frac{1}{2}}(Cz + D)^{\frac{1}{2}},$$

where A , B , C , and D are constants. Thus

$$\left. \begin{aligned} u &= -\frac{2C(Ar^2 + B)^{\frac{1}{2}}}{3r(Cz + D)^{\frac{1}{2}}} \\ w &= \frac{4A(Cz + D)^{\frac{1}{2}}}{3(Ar^2 + B)^{\frac{1}{2}}} \end{aligned} \right\}. \quad (6.4)$$

The stream lines are given by $\psi = \text{constant}$, i.e.

$$(Ar^2 + B)(Cz + D) = \text{constant}.$$

Compare this with section 3.1 (ii) (c), i.e. with the expression for ψ in (3.7). The stream lines are the same in both cases.

The velocity is infinite on the axis, and on the surfaces $\psi = 0$, owing to the singularity of the equation,

$$\left. \begin{aligned} H_1 &= (2\psi + 4\pi\rho)^{\frac{1}{2}}u, & H_2 &= 0, & H_3 &= (2\psi + 4\pi\rho)^{\frac{1}{2}}w \\ j_1 &= 0, & j_2 &= \frac{1}{4\pi}(H_{1z} - H_{3r}), & j_3 &= 0 \end{aligned} \right\}, \quad (6.5)$$

showing that the magnetic field is poloidal, but the current density field is toroidal.

Since the solution of the equations (6.3) when $m = 0$ is the $\frac{2}{3}$ rd power of the solution of the equation (3.4) in the same case (i.e. when $m = 0$), one may expect that this relationship between the two sets of equations exists when $m \neq 0$. In fact this is true if we assume $m = \frac{4}{3}n^2$ when $m > 0$, and $m = -\frac{4}{3}n^2$ when $m < 0$ in (6.3).

Hence the solution of (6.3) when $m = \frac{4}{3}n^2$ is

$$Z = (Ae^{nz} + Be^{-nz})^{\frac{1}{2}},$$

$$R = [rJ_1(nr)]^{\frac{1}{2}};$$

whence

$$\psi = [rJ_1(nr)(Ae^{nz} + Be^{-nz})]^{\frac{1}{2}},$$

and the components of velocity are

$$\left. \begin{aligned} u &= -\frac{2n[J_1(nr)]^{\frac{1}{2}}(Ae^{nz} - Be^{-nz})}{3r^{\frac{1}{2}}(Ae^{nz} + Be^{-nz})^{\frac{1}{2}}} \\ w &= \frac{2nJ_0(nr)(Ae^{nz} + Be^{-nz})^{\frac{1}{2}}}{3r^{\frac{1}{2}}[J_1(nr)]^{\frac{1}{2}}} \end{aligned} \right\}, \quad (6.6)$$

where J_1 , J_0 are Bessel's functions of order one and zero respectively. The solution when $m = -\frac{4}{3}n^2$ is

$$Z = (A \cos nz + B \sin nz)^{\frac{1}{2}},$$

$$R = [rI_1(nr)]^{\frac{1}{2}},$$

so that

$$\psi = [rI_1(nr)(A \cos nz + B \sin nz)]^{\frac{1}{2}}.$$

Hence

$$\left. \begin{aligned} u &= \frac{2n[I_1(nr)]^{\frac{1}{2}}(A \sin nz - B \cos nz)}{3r^{\frac{1}{2}}(A \cos nz + B \sin nz)^{\frac{1}{2}}} \\ w &= \frac{2nI_0(nr)(A \cos nz + B \sin nz)^{\frac{1}{2}}}{3r^{\frac{1}{2}}[I_1(nr)]^{\frac{1}{2}}} \end{aligned} \right\}, \quad (6.7)$$

where I_1 , I_0 are Bessel's functions of imaginary argument. In all cases (6.4), (6.6), and (6.7) the stream lines are given by $\psi = \text{constant}$. They are the same as in section 3.1 (ii). The magnetic and current density fields are found by (6.5).

7. A discussion of the singularity of the vorticity equation (2.10)

The coefficient of the highest derivative in the partial differential equation (2.10) is $A = g^2 - 4\pi\rho$. Hence, when $g^2 - 4\pi\rho = 0$ we may expect an infinity in the solution. In fact (2.11) shows that as $g^2 - 4\pi\rho \rightarrow 0$ both v and H_2 tend to infinity.

Consider the equations giving expressions for v and H_2 , namely,

$$4\pi\rho v - grH_2 = K, \quad (7.1)$$

$$grv - rH_2 = r^2f'. \quad (7.2)$$

If $g^2 - 4\pi\rho \neq 0$, these define rv and rH_2 as

$$rv = -\frac{K - r^2gf'}{g^2 - 4\pi\rho}, \quad (7.3)$$

$$rH_2 = -\frac{gK - 4\pi\rho r^2f'}{g^2 - 4\pi\rho}. \quad (7.4)$$

If $g^2 - 4\pi\rho = 0$ we cannot solve for rv and rH_2 , and the two equations (7.1) and (7.2) are only consistent if

$$K - r^2gf' = 0; \quad (7.5)$$

the two equations then become identical. Instead of the two relations, we can take the single relation (7.2), which can be written as

$$gv - H_2 = rf'. \quad (7.6)$$

The Bernoulli's equation (2.8) does not change, and the vorticity equation, if we take the form (2.9), becomes

$$4\pi\rho F' + \frac{gg'}{r^2}(\psi_r^2 + \psi_z^2) = \frac{1}{r}vK + vH_2g' - rH_2f''. \quad (7.7)$$

Using $g^2 - 4\pi\rho = 0$ and (7.5) it follows that

$$gK - 4\pi\rho r^2f' = 0;$$

hence both v and H_2 become indeterminate. We shall separate the two cases $g' = 0$ and $g' \neq 0$.

7.1. $g' = 0$

This means that g is an absolute constant, and the relation

$$g^2 - 4\pi\rho = 0 \quad (7.8)$$

is satisfied whatever the values of r and z , i.e. everywhere. In this case the vorticity equation (7.7) becomes

$$4\pi\rho F' = \frac{1}{r}vK' - rH_2f''. \quad (7.9)$$

The equations (7.6) and (7.9) are two equations to determine v and H_2 . If $K' - r^2gf'' \neq 0$, v and H_2 are

$$\left. \begin{aligned} v &= \frac{4\pi\rho F' - r^2f'f''}{K' - r^2gf''} r \\ H_2 &= \frac{4\pi\rho g F' - f'K'}{K' - r^2gf''} r \end{aligned} \right\} \quad (7.10)$$

Equation (7.5) shows that K and f' are either both zero or both different from zero. If $K = f' \equiv 0$, (7.10) does not define v and H_2 , since

$$K' - r^2gf'' = 0.$$

But since $f' = 0$, (7.6) shows that $gv - H_2 = 0$, that is $H_2 = \sqrt{(4\pi\rho)}v$ by (7.8). Hence it follows from (2.11) that

$$\mathbf{H} = \sqrt{(4\pi\rho)}\mathbf{V}. \quad (7.11)$$

This is the relation given by Chandrasekhar (3) as the simplest solution of the magnetohydrodynamic equations under the conditions considered. The equation (7.9) then gives $F = \text{constant}$; hence Bernoulli's equation (2.8) is

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = \text{constant}. \quad (7.12)$$

The results (7.11) and (7.12) follow directly from the main equations (2.6) and (2.7).

On the other hand, if one or other of K' or f'' is not zero, (7.5) indicates that ψ is a function of r^2 only. Then $u = H_1 = 0$, $w = (1/r)\psi_r$, $H_3 = (1/r)g_0\psi_r$, v , and H_2 are found by (7.10), where $g_0 = \sqrt{(4\pi\rho)}$.

If for example $K = g_0\psi^2$, $f' = a\psi$, $4\pi\rho F' = ab\psi$, where a, b are arbitrary constants; from (7.5) we get

$$g_0\psi(\psi - ar^2) = 0.$$

Hence either $\psi = 0$, or $\psi = ar^2$.

We cannot accept the case $\psi = 0$, since then $K = f' = 0$ and (7.10) does not define v and H_2 . When $\psi = ar^2$ we have

$$u = 0, \quad v = \frac{ar}{g_0}(b - ar^2), \quad w = 2a. \quad (7.13)$$

The stream lines are then

$$r = c_1, \quad z = \frac{2\sqrt{(4\pi\rho)}\theta}{b - ar^2} + c_2,$$

i.e. helices on coaxial cylinders, whose inclination to the generators of the cylinder is a function of r . Also

$$\left. \begin{aligned} H_1 &= 0, & H_2 &= ar(b - 2ar^2), & H_3 &= \sqrt{(4\pi\rho)}2a \\ j_1 &= 0, & j_2 &= 0, & j_3 &= \frac{a}{2\pi}(b - 4ar^2) \end{aligned} \right\}. \quad (7.14)$$

Bernoulli's equation gives p as a function of r .

7.2. $g' \neq 0$

If g is a function of ψ , g' is not generally zero, and the equation $g^2 - 4\pi\rho = 0$ is satisfied only on a surface of revolution defined by itself. Outside of this surface $g^2 - 4\pi\rho \neq 0$, and the normal equations (2.10) and (2.11) are used. On the singular surface $g^2 - 4\pi\rho = 0$, we have to use (7.5), (7.6), and (7.7) instead.

If K and f' are identically zero, v and H_2 will be zero everywhere. We can assume that this is true even on the surface of $g^2 - 4\pi\rho = 0$.

If $K - r^2 g f'$ is not identically zero, it must contain $g^2 - 4\pi\rho$ as a factor, since it must vanish when $g^2 - 4\pi\rho = 0$. Hence

$$K - r^2 g f' = \text{a multiple of } (g^2 - 4\pi\rho)^m,$$

and thence

$$\left. \begin{aligned} K &= \alpha(g^2 - 4\pi\rho)^m \\ f' &= \beta(g^2 - 4\pi\rho)^m \end{aligned} \right\} \quad (7.15)$$

where α and β are arbitrary functions of ψ which are finite when

$$g^2 - 4\pi\rho = 0,$$

and m is a positive arbitrary constant to be determined suitably.

This fact is realized in the general equations of sections 4 and 5. For example, the equation (4.1) suggests that both K^2 and f'^2 must be multiples of $g^2 - 4\pi\rho$. It is seen in (4.2) that if $f' \neq 0$, it must be a constant multiple of $g^2 - 4\pi\rho$. (4.3) shows that if $K = 0$, it is a multiple of $g^2 - 4\pi\rho = 0$. Again, the equations (5.1), (5.2), and (5.27) show that K and f' are either zero or multiples of $g^2 - 4\pi\rho$.

In any problem we usually know $g^2 - 4\pi\rho$ as a function of ψ , also we know the value of ψ in terms of r and z , i.e. the singular surface $g^2 - 4\pi\rho = 0$ is known. If on the singular surface the formulae (7.3) and (7.4) become indeterminate we use (7.6) and (7.7) to determine v and H_2 , under the conditions (7.15) imposed on K and f' .

Usually, if K and f' have the form (7.15) the formulae (7.3) and (7.4) will be determinate and can still be used to determine the values of v and H_2 . But in general the velocity is infinite on the singular surface. For example take the solution (4.8). If $K = \alpha(g^2 - 4\pi\rho)$, then $f' = 0$, and v will be finite on the singular surface but u is infinite. In (5.10), if f_r is chosen to be a multiple of $(g^2 - 4\pi\rho)^{\frac{1}{2}}$, $K = 0$, v becomes finite, but $w = \infty$ when $g^2 - 4\pi\rho = 0$. On the other hand, in (5.13) it is possible to choose f_r as a multiple of $g^2 - 4\pi\rho$, so that both u and w become finite on the singular surface, and everywhere if g is finite everywhere.

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FLOW OF A NON-NEWTONIAN LIQUID IN A CURVED PIPE

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SUMMARY

A theoretical analysis is made of the flow of an incompressible non-Newtonian viscous liquid in a curved pipe, looking for differences in observable characteristics from the corresponding case of Newtonian flow. Such a motion is of interest to the experimentalist because the flow could be readily attained and controlled in practice, the most easily measurable quantities being the axial pressure gradient and the volume rate of flow. It is assumed, for the purpose of mathematical analysis, that the curvature of the pipe is small, more precisely that the radius of the circle in which the central line of the pipe is coiled is large in comparison with the radius of the cross-section. A solution is developed by successive approximations, the first approximation corresponding to the flow of a Newtonian viscous liquid as given by Dean (1). The streamlines in the plane of symmetry and the projection of the streamlines on a normal section are compared with those of a Newtonian liquid.

1. Introduction

RIVLIN (2), (3) has proposed that the macroscopic behaviour of certain fairly dilute solutions of high polymers can be represented by equations of state in the form†

$$p_{ik} = -P\delta_{ik} + 2\eta e_{ik} + 4\zeta e_{ij}e_{jk}, \quad (1)$$

where the viscosity coefficient η and the normal-stress coefficient ζ are scalar functions of the flow invariants

$$I_1 \equiv e_{ii}, \quad I_2 \equiv \frac{1}{2}[(e_{ii})^2 - e_{ik}e_{ik}], \quad I_3 \equiv \det(e_{ik}). \quad (2)$$

Here p_{ik} and e_{ik} are the stress and rate-of-strain tensors, P is an undetermined isotropic pressure which can be superposed on an element of liquid (in any state of motion) without affecting the flow, and δ_{ik} is the Kronecker delta ($\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ if $i \neq k$). A formula of the form (1) in which the coefficients η and ζ are physical constants has been discussed by Reiner (4); Reiner considers the material to be compressible

† The usual convention of summation applies to repeated small Latin suffixes except r and z .

and accordingly the quantity P is a function of the density, i.e. of the dilatation.

A variety of interesting theoretical predictions concerning liquids characterized by the rheological equations of state (1) are revealed in some simple types of laminar flow discussed by Rivlin (2), (5) and by Braun and Reiner (6), results which are not a feature of the quasi-linear theory ($\zeta = 0$). In steady rectilinear flow in a straight pipe of circular cross-section the incompressibility condition $I_1 = 0$ is consistent with a velocity distribution $[0, 0, W(r)]$ when referred to cylindrical polar coordinates r, ϕ, z in which the z -axis coincides with the axis of the pipe. The equations of motion require (5)

$$\eta \frac{dW}{dr} = -\frac{1}{2}Ar, \quad P = \int \frac{1}{r} \frac{d}{dr} \left[\zeta r \left(\frac{dW}{dr} \right)^2 \right] dr - Az + B, \quad (3)$$

where η and ζ are in general functions of r , and A and B are arbitrary constants. The non-vanishing physical components of stress are then found to be

$$\left. \begin{aligned} \widehat{rr} = \widehat{zz} &= - \int \zeta \frac{1}{r} \left(\frac{dW}{dr} \right)^2 dr + Az - B \\ \widehat{\phi\phi} &= \int \frac{1}{r} \frac{d}{dr} \left[\zeta r \left(\frac{dW}{dr} \right)^2 \right] dr + Az - B, \quad \widehat{rz} = \eta \frac{dW}{dr} \end{aligned} \right\}. \quad (4)$$

It is seen from (3) and (4) that the axial gradient of \widehat{zz} is a constant A and that the differential equation for $W(r)$ is independent of the normal-stress coefficient ζ .

For reasons of mathematical convenience we shall assume throughout the ensuing discussion that the viscosity coefficient η and the normal-stress coefficient ζ are physical constants. Equations (3) reduce, in this case, to

$$W = \frac{A}{4\eta} (a^2 - r^2), \quad P = \zeta \frac{3A^2}{8\eta^2} r^2 - Az + B, \quad (5)$$

where we have taken $r = a$ to be the boundary of the pipe.

2. Flow in a curved pipe

We consider the steady motion of the liquids characterized by (1) through a pipe of circular cross-section, of radius a , with the line of centres coiled in a circle of radius b . The coordinate system used is that adopted by Dean (1) in discussing the flow of Newtonian liquids under the same conditions and is illustrated in Fig. 1. Ov is the axis of the circle in which the pipe is coiled, C is the centre of the section of the pipe by a plane

through $O\nu$ that makes an angle θ with a fixed axial plane, and CO is perpendicular to $O\nu$ and of length b . Any point P of the section $\theta = \text{constant}$ is referred to by the orthogonal curvilinear coordinates r, ϕ, θ , where

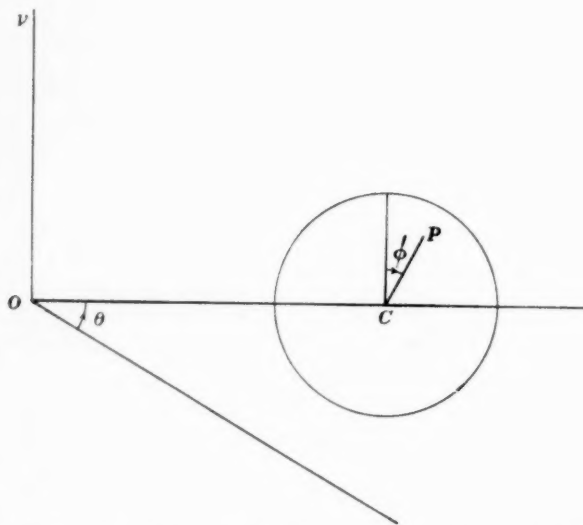


FIG. 1. The coordinate system (r, ϕ, θ) chosen to describe motion in a curved pipe of circular cross-section.

r is the distance CP and ϕ is the angle CP makes with the line through C parallel to $O\nu$; the line element is then given by

$$ds^2 = dr^2 + r^2 d\phi^2 + (b + r \sin \phi)^2 d\theta^2. \quad (6)$$

It is assumed that the motion is steady, and, further, that the velocity components in the coordinate directions r, ϕ, θ may be written $U(r, \phi)$, $V(r, \phi)$, $W(r, \phi)$, independent of θ . The rate-of-strain tensor has then the physical components

$$\left. \begin{aligned} e_{rr} &= \frac{\partial U}{\partial r}, & e_{\phi\phi} &= \frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{U}{r}, & e_{\theta\theta} &= \frac{U \sin \phi + V \cos \phi}{b + r \sin \phi} \\ e_{\phi\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b + r \sin \phi} \right), & e_{\theta r} &= \frac{1}{2} \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b + r \sin \phi} \right), \\ e_{r\phi} &= \frac{1}{2} \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right] \end{aligned} \right\}, \quad (7)$$

and, from (1), the stress tensor has the physical components

$$\begin{aligned}
 \widehat{rr} &= -P + 2\eta \frac{\partial U}{\partial r} + \zeta \left\{ 4 \left(\frac{\partial U}{\partial r} \right)^2 + \right. \\
 &\quad \left. + \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right]^2 + \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right)^2 \right\} \\
 \widehat{\phi\phi} &= -P + 2\eta \left(\frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{U}{r} \right) + \zeta \left\{ 4 \left(\frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{U}{r} \right)^2 + \right. \\
 &\quad \left. + \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right]^2 + \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right)^2 \right\} \\
 \widehat{\theta\theta} &= -P + 2\eta \frac{U \sin \phi + V \cos \phi}{b+r \sin \phi} + \zeta \left\{ 4 \left(\frac{U \sin \phi + V \cos \phi}{b+r \sin \phi} \right)^2 + \right. \\
 &\quad \left. + \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right)^2 + \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right)^2 \right\} \\
 \widehat{\phi\theta} &= \eta \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right) + \\
 &\quad + \zeta \left\{ 2 \left(\frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{U}{r} + \frac{U \sin \phi + V \cos \phi}{b+r \sin \phi} \right) \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right) + \right. \\
 &\quad \left. + \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right] \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right) \right\} \\
 \widehat{\theta r} &= \eta \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right) + \\
 &\quad + \zeta \left\{ 2 \left(\frac{\partial U}{\partial r} + \frac{U \sin \phi + V \cos \phi}{b+r \sin \phi} \right) \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right) + \right. \\
 &\quad \left. + \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right] \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right) \right\} \\
 \widehat{r\phi} &= \eta \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right] + \zeta \left\{ 2 \left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{r} \frac{\partial V}{\partial \phi} \right) \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) + \frac{1}{r} \frac{\partial U}{\partial \phi} \right] + \right. \\
 &\quad \left. + \left(\frac{\partial W}{\partial r} - \frac{W \sin \phi}{b+r \sin \phi} \right) \left(\frac{1}{r} \frac{\partial W}{\partial \phi} - \frac{W \cos \phi}{b+r \sin \phi} \right) \right\}
 \end{aligned} \tag{8}$$

The stress equations of motion require, in the absence of body forces,

$$\rho \left(U \frac{\partial U}{\partial r} + \frac{V}{r} \frac{\partial U}{\partial \phi} - \frac{V^2}{r} - \frac{W^2 \sin \phi}{b+r \sin \phi} \right) = \frac{\partial \widehat{rr}}{\partial r} + \frac{\widehat{rr} - \widehat{\phi\phi}}{r} + \frac{\widehat{r\phi} \cos \phi}{b+r \sin \phi} + \frac{1}{r} \frac{\partial \widehat{r\phi}}{\partial \phi} + \frac{(\widehat{rr} - \widehat{\theta\theta}) \sin \phi}{b+r \sin \phi} + \frac{1}{b+r \sin \phi} \frac{\partial \widehat{\theta r}}{\partial \theta}, \quad (9)$$

$$\rho \left(U \frac{\partial V}{\partial r} + \frac{V}{r} \frac{\partial V}{\partial \phi} + \frac{UV}{r} - \frac{W^2 \cos \phi}{b+r \sin \phi} \right) = \frac{\partial \widehat{r\phi}}{\partial r} + \left(\frac{2}{r} + \frac{\sin \phi}{b+r \sin \phi} \right) \widehat{r\phi} + \frac{1}{r} \frac{\partial \widehat{\phi\phi}}{\partial \phi} + \frac{(\widehat{\phi\phi} - \widehat{\theta\theta}) \cos \phi}{b+r \sin \phi} + \frac{1}{b+r \sin \phi} \frac{\partial \widehat{\phi\theta}}{\partial \theta}, \quad (10)$$

$$\rho \left(U \frac{\partial W}{\partial r} + \frac{V}{r} \frac{\partial W}{\partial \phi} + \frac{UW \sin \phi}{b+r \sin \phi} + \frac{VW \cos \phi}{b+r \sin \phi} \right) = \frac{\partial \widehat{\theta r}}{\partial r} + \left(\frac{1}{r} + \frac{2 \sin \phi}{b+r \sin \phi} \right) \widehat{\theta r} + \frac{1}{r} \frac{\partial \widehat{\phi\theta}}{\partial \phi} + \frac{2 \widehat{\phi\theta} \cos \phi}{b+r \sin \phi} + \frac{1}{b+r \sin \phi} \frac{\partial \widehat{\theta\theta}}{\partial \theta}. \quad (11)$$

The equation of continuity, $I_1 = 0$, requires

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{U \sin \phi}{b+r \sin \phi} + \frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{V \cos \phi}{b+r \sin \phi} = 0. \quad (12)$$

If the pipe were straight a/b would vanish, and equations (8)–(12) would be satisfied by the expressions (5) for W and P together with $U = 0$, $V = 0$.

We now assume that the curvature of the pipe is small, that is, a/b is small. Following Dean (1) we write

$$U = u, \quad V = v, \quad W = \frac{A}{4\eta} (a^2 - r^2) + w, \quad P = \zeta \frac{3A^2}{8\eta^2} r^2 - Az + B + p, \quad (13)$$

where u , v , w , and p are all taken to be small, of the order of a/b , and u , v , w are independent of θ . Neglecting terms of order $(a/b)^2$ the equation of continuity becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \phi} = 0, \quad (14)$$

and is the condition for the existence of a stream function ψ such that

$$u = -\frac{1}{r} \frac{\partial \psi}{\partial \phi}, \quad v = \frac{\partial \psi}{\partial r}. \quad (15)$$

The stress equations of motion (9)–(11) now reduce to

$$-\frac{\rho A^2}{16\eta^2 b} (a^2 - r^2)^2 \sin \phi - \frac{\zeta A^2}{8\eta^2 b} (3a^2 - 7r^2) \sin \phi$$

$$= -\frac{\partial p}{\partial r} - \eta \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} \right) - \frac{\zeta A}{2\eta} \left(2r \frac{\partial^2 w}{\partial r^2} + 4 \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \phi^2} \right), \quad (16)$$

$$-\frac{\rho A^2}{16\eta^2 b} (a^2 - r^2)^2 \cos \phi - \frac{\zeta A^2}{8\eta^2 b} (3a^2 - 7r^2) \cos \phi$$

$$= -\frac{1}{r} \frac{\partial p}{\partial \phi} + \eta \frac{\partial}{\partial r} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} \right) - \frac{\zeta A}{2\eta} \frac{\partial}{\partial \phi} \left(\frac{\partial w}{\partial r} + 2 \frac{w}{r} \right), \quad (17)$$

$$\frac{\rho A}{2\eta} \frac{\partial \psi}{\partial \phi} + \frac{3A}{2b} r \sin \phi$$

$$= -\frac{\partial p}{\partial z} + \eta \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} \right) + \frac{\zeta A}{2\eta} \frac{\partial}{\partial \phi} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} \right), \quad (18)$$

where we have written, in conformity with (13), $z = b\theta$ and $(1/b)\partial/\partial\theta = \partial/\partial z$.

It will be seen that there exists a solution of equations (16), (17), and (18) in the form

$$\psi = \frac{Aa^4}{4\eta b} \psi' \left(\frac{r}{a} \right) \cos \phi, \quad w = \frac{Aa^3}{4\eta b} w' \left(\frac{r}{a} \right) \sin \phi, \quad p = \frac{Aa^2}{4b} p' \left(\frac{r}{a} \right) \sin \phi, \quad (19)$$

where ψ' , w' , and p' are functions of r/a only. Substituting for ψ , w , and p in terms of the dimensionless quantities ψ' , w' , and p' in (16)–(18), the following equations are obtained:

$$-R(1-r'^2)^2 - 2S(3-7r'^2) = -\frac{dp'}{dr'} + \frac{1}{r'} \Delta \psi' - 2S \left(2r' \frac{d^2 w'}{dr'^2} + 4 \frac{dw'}{dr'} - \frac{w'}{r'} \right), \quad (20)$$

$$-R(1-r'^2)^2 - 2S(3-7r'^2) = -\frac{p'}{r'} + \frac{d}{dr'} \Delta \psi' - 2S \left(\frac{dw'}{dr'} + 2 \frac{w'}{r'} \right), \quad (21)$$

$$-2R\psi' + 6r' = \Delta w' - 2S\Delta \psi', \quad (22)$$

where R (a Reynolds number) and S are the dimensionless parameters

$$R = \frac{\rho a^3 A}{4\eta^2}, \quad S = \frac{\zeta a A}{4\eta^2}, \quad (23)$$

Δ is the operator defined by

$$\Delta \equiv \frac{d^2}{dr'^2} + \frac{1}{r'} \frac{d}{dr'} - \frac{1}{r'^2}, \quad (24)$$

and r' is written for r/a . The condition that there is no slip at the boundary of the pipe is satisfied if

$$\psi' = \frac{d\psi'}{dr'} = w' = 0 \quad \text{on } r' = 1. \quad (25)$$

Also ψ'/r' , $d\psi'/dr'$ and w' must be finite at all points in the region

$$0 \leq r' \leq 1.$$

Dean (1) has shown, in the case when $S = 0$, that a necessary condition for a solution of physical significance is

$$\frac{\psi'}{r'} = \frac{d\psi'}{dr'} \quad \text{at } r' = 0, \quad (26)$$

$$\text{and} \quad w' = p' = 0 \quad \text{at } r' = 0, \quad (27)$$

and his argument applies generally when $S \neq 0$.

Eliminating p' from equations (20) and (21) we obtain

$$\Delta(\Delta\psi' + 2Sw') = 4R(1-r'^2)r' + 28Sr'. \quad (28)$$

Comparing equations (22) and (28) it is seen that w' may be eliminated quite easily; we find that

$$\Delta^2\psi' + 4S^2\Delta\psi' - 4RS\psi' = 4Rr'(1-r'^2) + 16Sr', \quad (29)$$

and this equation for ψ' is to be solved subject to the boundary conditions (25) and (26). When S is sufficiently small a solution of (25), (26), and (29) can be developed with successive approximations for ψ' equal to the partial sums of a series

$$\psi' = \psi_0 + S\psi_1 + S^2\psi_2 + S^3\psi_3 + \dots \quad (30)$$

in the following way. The series (30) is substituted in (29), and the left-hand side is expanded in ascending powers of S . Equating in turn the coefficients of S^0 , S^1 , S^2 , ... differential equations for ψ_0 , ψ_1 , ψ_2 , ... are obtained. The process of substitution and equating coefficients yields the following equations:

$$\Delta^2\psi_0 = 4Rr'(1-r'^2), \quad (31)$$

$$\Delta^2\psi_1 = 4R\psi_0 + 16r', \quad (32)$$

$$\Delta^2\psi_n = 4R\psi_{n-1} - 4\Delta\psi_{n-2} \quad (n \geq 2). \quad (33)$$

Each successive approximation to ψ' must satisfy the boundary conditions, and this requires

$$\psi_n = \frac{d\psi_n}{dr'} = 0 \quad (n \geq 0) \quad \text{on } r' = 1, \quad (34)$$

$$\text{and} \quad \frac{\psi_n}{r'} = \frac{d\psi_n}{dr'} \quad (n \geq 0) \quad \text{at } r' = 0; \quad (35)$$

also ψ_n/r' and $d\psi_n/dr'$ (all n) must remain finite in the region $0 \leq r' \leq 1$.

The first approximation ψ_0 is found to be

$$\psi_0 = r'(1-r'^2)^2(4-r'^2) \frac{R}{288}, \quad (36)$$

in agreement with Dean (1). Substituting this value of ψ_0 in equation (32) we obtain

$$\Delta^2 \psi_1 = r'(1-r'^2)^2(4-r'^2) \frac{R^2}{72} + 16r', \quad (37)$$

the solution of which is

$$\psi_1 = Kr' + Lr' \log r' + \frac{M}{r'} + Nr'^3 + r'^5 \left(\frac{1}{3} - \frac{1}{8}r'^2 + \frac{1}{40}r'^4 - \frac{1}{600}r'^6 \right) \frac{R^2}{1152} + \frac{1}{12}r'^5, \quad (38)$$

where K , L , M , and N are constants. The boundary conditions (34) and (35) require

$$\left. \begin{aligned} L &= M = 0 \\ K + N + \frac{139R^2}{691200} + \frac{1}{12} &= 0 \\ K + 3N + \frac{599R^2}{691200} + \frac{5}{12} &= 0 \end{aligned} \right\}. \quad (39)$$

The resulting expression for ψ_1 is

$$\psi_1 = r'(1-r'^2)^2 \left[(91 - 48r'^2 + 13r'^4 - r'^6) \frac{R^2}{691200} + \frac{1}{12} \right]. \quad (40)$$

There is no difficulty, in principle, in finding the remaining coefficients ψ_2 , ψ_3 , etc., but the algebra becomes rather heavy. For the purpose of illustration we shall now assume that to a sufficient approximation ψ' may be taken as a linear function in S .

By inspection of equations (20) to (22) it follows that we may write

$$p' = p_0 + Sp_1 + S^2p_2 + S^3p_3 + \dots \quad (41)$$

and
$$w' = w_0 + Sw_1 + S^2w_2 + S^3w_3 + \dots \quad (42)$$

The boundary conditions (25) and (27) require

$$w_n = 0 \quad (\text{all } n) \quad \text{on } r' = 1, \quad (43)$$

and
$$w_n = p_n = 0 \quad (\text{all } n) \quad \text{at } r' = 0; \quad (44)$$

also w_n (all n) must remain finite in the region $0 \leq r' \leq 1$. Substituting (30) and (42) in the differential equation (22), and equating in turn the coefficients of S^n ($n = 0, 1, 2, \dots$) on both sides of the equation, we obtain

$$\Delta w_0 = 6r' - 2R\psi_0, \quad (45)$$

$$\Delta w_n = -2R\psi_n + 2\Delta\psi_{n-1} \quad (n \geq 1), \quad (46)$$

where ψ_0, ψ_1, \dots are already determined. Substituting for ψ_0 and ψ_1 in (45) and (46) the first two coefficients of the series (42) satisfy

$$\Delta w_0 = 6r' - r'(1-r'^2)^2(4-r'^2) \frac{R^2}{144}, \quad (47)$$

and

$$\Delta w_1 = -r'(4-8r'^2+3r'^4)\frac{1}{6}R - \frac{r'(1-r'^2)^2(91-48r'^2+13r'^4-r'^6)R^3}{345600}. \quad (48)$$

The solution of (47) satisfying the conditions (43) and (44) has been found by Dean (1) to be

$$w_0 = -\frac{3}{4}r'(1-r'^2)+r'(1-r'^2)(19-21r'^2+9r'^4-r'^6)\frac{R^2}{11520}. \quad (49)$$

Integrating (48) under the conditions (43) and (44) we obtain

$$w_1 = r'(1-r'^2)(11-13r'^2+3r'^4)\frac{R}{288} - \frac{r'(1-r'^2)(1727-2095r'^2+1125r'^4-275r'^6+40r'^8-2r'^{10})R^3}{116121600}. \quad (50)$$

Substituting the series (30), (41), and (42) into (21) and equating in turn the coefficients S^n ($n = 0, 1, 2, \dots$) we find

$$p_0 = r'(1-r'^2)^2R + r'\frac{d}{dr'}\Delta\psi_0, \quad (51)$$

$$p_1 = 2r'(3-7r'^2)+r'\frac{d}{dr'}\Delta\psi_1 - 2r'\left(\frac{dw_0}{dr'} + \frac{2}{r'}w_0\right), \quad (52)$$

$$p_n = r'\frac{d}{dr'}\Delta\psi_n - 2r'\left(\frac{dw_{n-1}}{dr'} + \frac{2}{r'}w_{n-1}\right) \quad (n \geq 2). \quad (53)$$

The first two coefficients are found to be

$$p_0 = r'(9-6r'^2+2r'^4)\frac{R}{12}, \quad (54)$$

$$p_1 = \frac{1}{6}r'(55-93r'^2)+r'\left(-\frac{217}{15}+64r'^2-72r'^4+32r'^6-4r'^8\right)\frac{R^2}{1152}. \quad (55)$$

There is no difficulty in principle in proceeding to evaluate the remaining coefficients in the expansions (41) and (42), but it will suffice for illustration purposes to neglect terms of order S^2 .

What is perhaps of most interest is the effect of the normal-stress coefficient ζ on the relation between the axial pressure gradient and the rate of outflow. The rate of outflow is

$$\int_{r'=0}^{r'=1} \int_{\phi=0}^{\phi=2\pi} a^2 r' W \, dr' d\phi$$

and, to the above order of approximations, is independent of w' (and therefore of ζ), so that the relation between axial pressure gradient and rate of outflow is the same as if the pipe were straight. Dean (7), in a later

paper, has shown that this relation will be modified if terms $O(a/b)^2$ are included. Thus, to determine how the rate of flow depends on the normal stress coefficient ζ , for a given axial pressure gradient, we should have to consider the terms $O(a/b)^2$; this has so far proved too difficult.

The flow lines will vary with the dimensionless parameter S in a way that can readily be demonstrated even when terms $O(a/b)^2$ are neglected. The differential equation for the streamline in the central plane is

$$\frac{dr}{U} = \frac{(b+r \sin \phi) d\theta}{W} \quad (\phi = \pm \frac{1}{2}\pi), \quad (56)$$

and to a sufficient approximation we may write (56) as

$$\pm \frac{d\theta}{dr} = \frac{72}{R(1-r'^2)(1-\frac{1}{4}r'^2)} - \frac{72S}{R(1-r'^2)(1-\frac{1}{4}r'^2)^2} \times \\ \times \left\{ \frac{6}{R} + \frac{(91-48r'^2+13r'^4-r'^6)R}{9600} \right\}, \quad (57)$$

the sign \pm being the sign of ϕ . It follows, on integration of (57), that the equations of the streamlines are

$$\pm \theta = \theta_0 + S\theta_1, \quad (58)$$

$$\theta_0 = \frac{24}{R} \log \left[\frac{(1+r')^2(1-\frac{1}{2}r')}{(1-r')^2(1+\frac{1}{2}r')} \right], \quad (59)$$

$$\theta_1 = -\frac{3}{25}r' + \left(\frac{72}{R^2} + \frac{43}{800} \right) \frac{r'}{1-\frac{1}{4}r'^2} - \\ - \left(\frac{384}{R^2} + \frac{11}{30} \right) \log \left(\frac{1+r'}{1-r'} \right) + \left(\frac{264}{R^2} + \frac{281}{2400} \right) \log \left(\frac{1+\frac{1}{2}r'}{1-\frac{1}{2}r'} \right), \quad (60)$$

θ being measured from the point where the streamline crosses the central line ($r' = 0$). The value of θ_0 is in agreement with that of Dean (1), corresponding as it does to the value of θ when $S = 0$ (a Newtonian liquid). The (θ, r') relation is seen to be independent of the ratio a/b .

For a given value of r' , θ varies with the dimensionless parameters R and S ; in the case of a Newtonian liquid ($S = 0$) θ varies inversely as R . The value of θ_0 increases steadily with r' and tends to infinity as r' tends to unity, while θ_1 decreases steadily and tends to minus infinity as r' tends to unity. However, in the range of values of S for which the approximations made in this paper are valid, the function $\theta_0 + S\theta_1$ tends to infinity as r' tends to unity; the speed at which this function approaches infinity increases as S decreases. This shows, quite generally, that the curvature of the streamlines in the central plane increases (so that they follow more nearly the line of the pipe) as S decreases. Numerical illustrations are

now given for the particular boundary and Reynolds number considered by Dean (1), namely,

$$R = 63.3, \quad a/b = \frac{1}{3}, \quad (61)$$

and for different values of the parameter S . (As Dean (1) has pointed out, the value one-third for the ratio a/b is rather large and it is doubtful whether the approximations made in the above discussion are strictly valid for this value of a/b .) The values θ_0 and θ_1 for corresponding values of r' are given in the Table. It is seen that the contribution of θ_1 to θ is

Values of θ_0 and θ_1 in degrees for corresponding values of r'

r'	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
θ_0	0	6.6	13.3	20.3	28.0	36.6	46.8	59.5	77.0	89.4	106.8
θ_1	0	-4.5	-9.6	-14.9	-20.5	-26.7	-33.9	-42.7	-54.4	-62.3	-73.5

positive when $S < 0$ and negative when $S > 0$. There is a restriction on the magnitude of S , if the expressions for ψ' , w' , and p' are to be convergent. It is assumed that the series (30), (41), and (42) are all convergent when $|S|$ is less than about 1.5, as seems likely from the first few terms. The forms of the whole streamlines for the particular values $S = -1.0$, $S = 0$, and $S = 1.0$ are shown in Figs. 2, 3, and 4, respectively. The effect of a positive S (a liquid which can exhibit the positive Weissenberg effect) is to decrease the angular distance that the fluid particles in the central plane travel in going from points near the inner edge of the pipe to points near the outer edge; the effect of a negative S (a liquid capable of exhibiting the less frequently observed negative Weissenberg effect) is to increase this angular distance. The values of S have been chosen rather large in order to demonstrate the qualitative effect of non-Newtonian behaviour; in an accurate solution we should, of course, have to include more terms in the series expansion.

It is of interest to draw also the curves of intersection of the surfaces $\psi = \text{constant}$ with a normal section $\theta = \text{constant}$. Dean (1) has done this in the case of a Newtonian liquid ($S = 0$). The curves have the polar equation

$$\sec \phi = kr'(1-r'^2)^2(1-\frac{1}{4}r'^2) \times$$

$$\times \left[1 + (1 - \frac{1}{4}r'^2)^{-1} \left\{ (91 - 48r'^2 + 13r'^4 - r'^6) \frac{R}{9600} + \frac{6}{R} \right\} S \right], \quad (62)$$

where k is an arbitrary constant. When $S = 0$ the solution is in agreement with Dean (1) and the relation between r' and ϕ in this case depends neither on a/b nor on R . For the class of liquids considered here it is seen

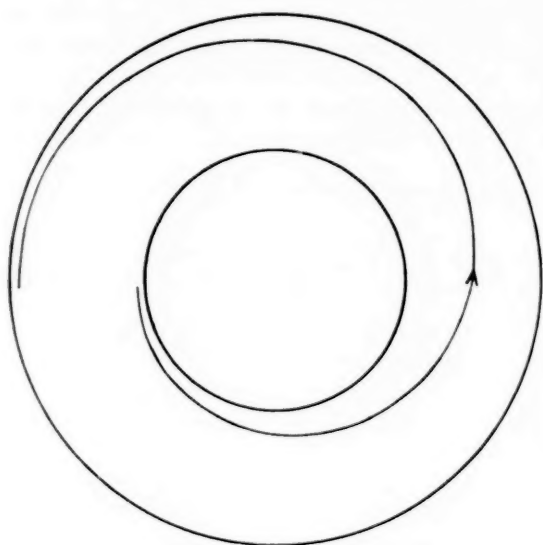


FIG. 2. Streamline in the central plane when $S = -1.0$ and $R = 63.3$.

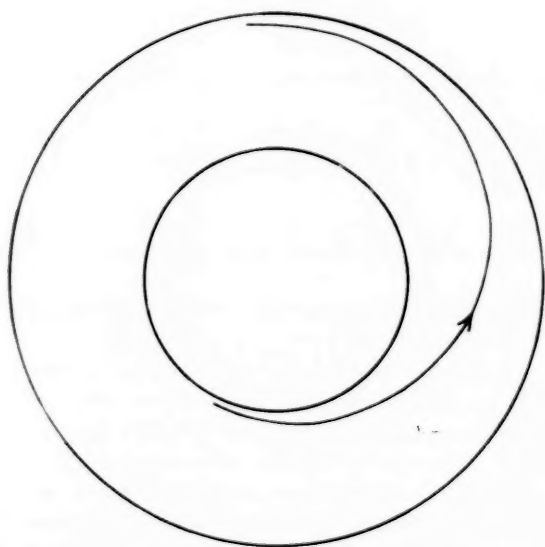


FIG. 3. Streamline in the central plane when $S = 0$ and $R = 63.3$.

that for any given value of r' the value of $\sec \phi$ varies with the Reynolds number R and with the parameter S ; again the (r, ϕ) relation is independent of the ratio a/b . We shall take the Reynolds number to be fixed and to have the value 63.3, and consider the effect of a variation in S .

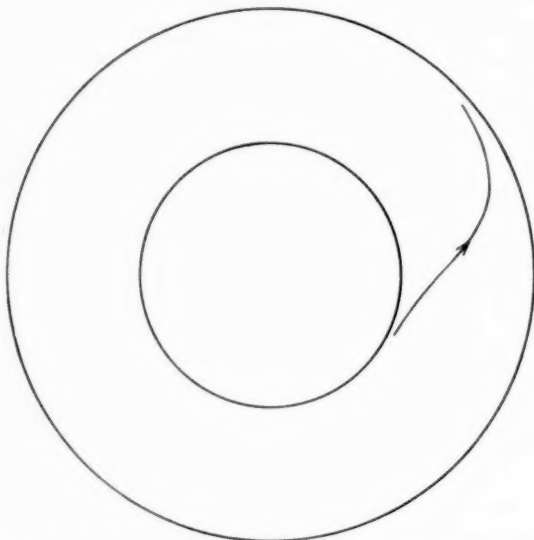


FIG. 4. Streamline in the central plane when $S = 1.0$ and $R = 63.3$.

From equations (13), (15), (19), (30), (36), and (40) we see that to a sufficient approximation V vanishes, for all values of ϕ , when

$$4 - 23r'^2 + 7r'^4 + S \left[(91 - 599r'^2 + 401r'^4 - 124r'^6 + 11r'^8) \frac{R}{2400} + (1 - 5r'^2) \frac{24}{R} \right] = 0. \quad (63)$$

In the case of a Newtonian liquid ($S = 0$) the only relevant solution is $r' = 0.43$ (in agreement with Dean (1)) independently of the Reynolds number R . Taking $R = 63.3$ the relevant solutions of (63), in the particular cases $S = -1.4$ and $S = 1.4$, are $r' = 0.58$ and $r' = 0.42$, respectively. Thus at the points $r' = 0.58$, $\phi = n\pi$ when $S = -1.4$, $r' = 0.43$, $\phi = n\pi$ when $S = 0$ and $r' = 0.42$, $\phi = n\pi$ when $S = 1.4$, both U and V vanish, and there is a particular streamline in the form of a circle $r' = \text{constant}$, $\phi = n\pi$. For any particular value of S , there is therefore a limiting surface $\psi = \psi_l$ (the constant ψ_l dependent on S) which takes the degenerate form of a single circular streamline in a plane parallel to the

central plane. The intersection of the streamlines $\psi = \psi_l$ with a section $\theta = \text{constant}$ are denoted by dots in Figs. 5, 6, and 7. The line $\psi = \psi_l$ is defined by $k = 49.8$ when $S = -1.4$, by $k = 3.7$ when $S = 0$ and by $k = 1.9$ when $S = 1.4$; $k = \infty$, for all values of S , corresponds to the pipe wall. Denoting the value of k corresponding to $\psi = \psi_l$ by k_l , the surfaces

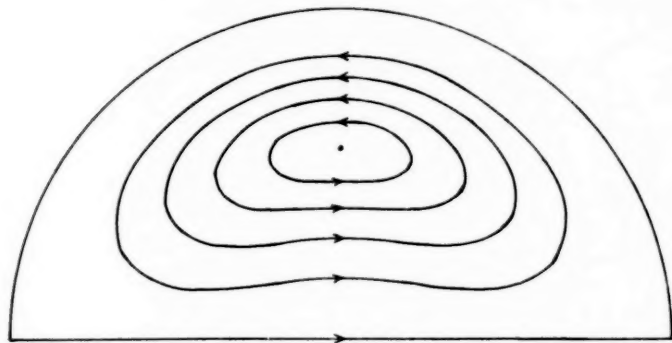


FIG. 5. Curves of intersection of the surfaces $\psi = \text{constant}$ with a normal section $\theta = \text{constant}$ when $S = -1.4$ and $R = 63.3$.

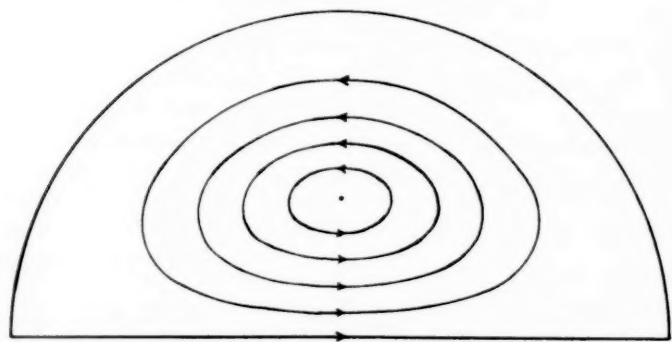


FIG. 6. Curves of intersection of the surfaces $\psi = \text{constant}$ with a normal section $\theta = \text{constant}$ when $S = 0$.

corresponding to $k/k_l = 1, 1.1, 1.3, 1.8, 3.5$ and ∞ (along the pipe wall) are indicated in Figs. 5, 6, and 7. It is seen that as S increases from -1.4 to 1.4 the distance from the central plane of the limiting streamline $\psi = \psi_l$ decreases from $r' = 0.58$ to $r' = 0.42$, an effect which might be clearly observable in an experiment; the general tendency is for the surfaces $\psi = \text{constant}$ to crowd nearer the limiting circular streamline as S decreases.

We have already shown that the angular distance (θ), in which the projection of the fluid element describes part of the closed curves such as those in Figs. 5, 6, and 7, decreases with the parameter S for a fixed

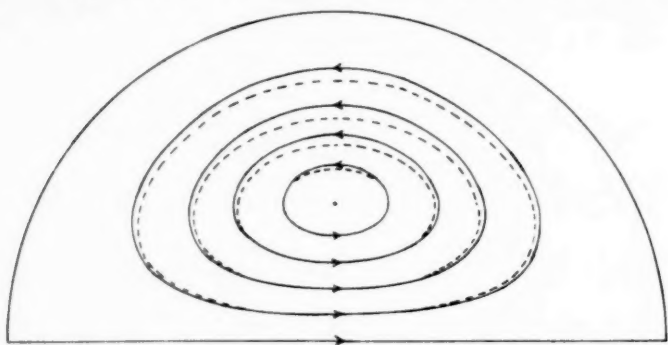


FIG. 7. Curves of intersection of the surfaces $\psi = \text{constant}$ with a normal section $\theta = \text{constant}$ when $S = 1.4$ and $R = 63.3$. Broken lines indicate the corresponding curves for a Newtonian liquid ($S = 0$).

Reynolds number R . This may be shown also from the (ϕ, θ) relation. To a sufficient approximation we find that

$$\frac{d\theta}{d\phi} \cos \phi = \frac{288r'}{R(4-23r'^2+7r'^4)} - \frac{288Sr'}{R(4-23r'^2+7r'^4)^2} \times \left\{ (91-599r'^2+401r'^4-124r'^6+11r'^8) \frac{R}{2400} + \frac{24}{R}(1-5r'^2) \right\}, \quad (64)$$

where the relation between r' and ϕ is known. For points very near the boundary we can write $r' = 1$ approximately and (64) becomes

$$\frac{d\theta}{d\phi} = (-0.38 + 0.23S) \sec \phi. \quad (65)$$

The angular distance in which a fluid particle near the boundary goes from a point where $\phi = \alpha$ to $\phi = 0$ is given by

$$[\theta]_{\phi=\alpha}^{\phi=0} = \int_{\alpha}^0 (-0.38 + 0.23S) \sec \phi \, d\phi = (0.38 - 0.23S) \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha). \quad (66)$$

This shows plainly that $[\theta]_{\phi=\alpha}^{\phi=0}$ increases as the parameter S decreases, and tends to infinity, whatever the value of S in the range $|S| < 1.5$, say, as α tends to $\frac{1}{2}\pi$; these results are compatible with those derived in the earlier part of the discussion.

It is of interest to pursue the problem further to illustrate the kinds of stress distributions which arise in non-Newtonian flow. The stresses acting across the bounding surface ($r' = 1$) are $[-\hat{r}\hat{r}]_{r'=1}$ and $[\hat{r}\hat{z}]_{r'=1}$.

Neglecting terms of order $(a/b)^2$ we obtain

$$\widehat{r}\widehat{r} - \widehat{r}\widehat{r}_0 = \left[-p' + 2\left(\frac{1}{r'}\frac{d\psi'}{dr'} - \frac{\psi'}{r'^2}\right) - 4Sr'\frac{dw'}{dr'} + 4Sr'(1-r'^2) \right] \frac{Aa^2}{4b} \sin \phi \quad (67)$$

and
$$\widehat{r}\widehat{z} - \widehat{r}\widehat{z}_0 = \left[\frac{dw'}{dr'} - 1 + r'^2 - 4S\left(\frac{d\psi'}{dr'} - \frac{\psi'}{r'}\right) \right] \frac{Aa^2}{4b} \sin \phi, \quad (68)$$

where $\widehat{r}\widehat{r}_0$ and $\widehat{r}\widehat{z}_0$ correspond to the values of $\widehat{r}\widehat{r}$ and $\widehat{r}\widehat{z}$, respectively, when a/b vanishes (see equations (4)) and are functions of r and z only. If δD is the axial drag on a small axial length δz of the cylinder then

$$\delta D = \int_{\phi=0}^{\phi=2\pi} \widehat{r}\widehat{z} a \, d\phi \delta z = 2\pi a \widehat{r}\widehat{z}_0 \delta z, \quad (69)$$

exactly as for a straight pipe except that here the direction of δD varies with the axial distance z .

From (67), we find finally

$$-(\widehat{r}\widehat{r} - \widehat{r}\widehat{r}_0)_{r'=1} = \left[\frac{5}{12}R - \left(\frac{1}{3} - \frac{11R^2}{17280}\right)S \right] \frac{Aa^2}{4b} \sin \phi. \quad (70)$$

Taking the Reynolds number R to be fixed and to have the value 63.3, (70) reduces to

$$-(\widehat{r}\widehat{r} - \widehat{r}\widehat{r}_0)_{r'=1} = (26.38 + 2.22S) \frac{Aa^2}{4b} \sin \phi.$$

This result shows that the effect of a negative S is to set up normal pressures which tend to keep the boundary section circular and that a positive S tends to make the pipe wall collapse.

Eustice (8) has made studies of the flow of Newtonian liquids (in this case water) in pipes of various radii of curvature. Similar experiments with different non-Newtonian liquids do not appear to have been carried out, so that the theoretical predictions cannot at this stage be subjected to any observational test.

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THE MOTION OF AN ELASTICO-VISCOUS LIQUID CONTAINED BETWEEN COAXIAL CYLINDERS (II)

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SUMMARY

A new representation of the relaxation spectrum of a liquid is used in order to develop the theory of oscillatory flow of the most general linear elastico-viscous liquid in a coaxial-cylinder elastoviscometer of the type built by Oldroyd, Strawbridge, and Toms. It is shown that experimental results concerning dilute polymer solutions, hitherto interpreted in terms of a discrete relaxation spectrum, can be alternatively interpreted in terms of a simple continuous relaxation spectrum characterized by three constants. It is also shown that the ambiguities of interpretation arising from experiments in which the amplitude ratio alone is measured as a function of frequency could in principle be partly resolved by measuring in addition the phase-lag between the inner and outer cylinders of the apparatus over the same frequency range.

1. Introduction

IN the first paper of the same title (1), consideration was given to the theory of flow of one particular elastico-viscous liquid prototype in a coaxial cylinder elastoviscometer of the type built by Oldroyd, Strawbridge, and Toms (2). The prototype considered was one which has proved very useful in characterizing certain dilute polymer solutions (2, 3, 4), namely, that represented by an equation of state of the form

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p'_{ik} = 2\eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e_{ik}^{(1)}, \quad (1)$$

at sufficiently small rates of strain. The notation is that of (1), $e_{ik}^{(1)}$ being the rate of strain tensor and p'_{ik} that part of the stress tensor associated with change of shape of the material element. In the present paper, the earlier work is extended to include the most general linear elastico-viscous liquid. This involves a consideration of the relaxation spectrum and in order to deal with mobile liquid systems satisfactorily it is found convenient to consider a somewhat different representation of the spectrum from those found in the literature (see, for example, 5, 6, 7).

Theoretical predictions are made on the basis of a number of idealized continuous relaxation spectra, and it is shown that experimental results

concerning dilute polymer solutions, hitherto interpreted in terms of a discrete relaxation spectrum corresponding to equation (1), can be alternatively interpreted in terms of a continuous relaxation spectrum of a simple type requiring no more physical constants to specify it. The experimental method of measuring amplitude ratio alone in an elastoviscometer is in fact insensitive even to certain gross changes in the form of the spectrum. It is shown that the method can be made sensitive by measuring in addition the phase-lag between the cylinders.

2. The relaxation spectrum

In order to describe the relaxation spectrum in the most appropriate way, we recall first the simplest (Maxwell) model of an elastico-viscous liquid whose behaviour is characterized by an equation of state of the form (cf. 7)

$$\frac{p'_{ik}}{\eta} + \frac{\dot{p}'_{ik}}{\zeta} = 2e_{ik}^{(1)}, \quad (2)$$

or, what is equivalent,

$$p'_{ik} = \frac{2\eta}{\tau} \int_{-\infty}^t e^{-(t-t')\tau} e_{ik}^{(1)}(t') dt', \quad (3)$$

where $\tau (= \eta/\zeta)$ is the relaxation time. Considering next a number of discrete Maxwell elements connected in parallel in the conventional spring-dashpot model representation, we can generalize (3) with the aid of the superposition principle (8), to give

$$p'_{ik} = \sum_{i=1}^n \frac{2\eta_i}{\tau_i} \int_{-\infty}^t e^{-(t-t')\tau_i} e_{ik}^{(1)}(t') dt', \quad (4)$$

where η_i and τ_i correspond to the i th Maxwell element.

The theoretical extension to a continuous distribution of relaxation times has been considered by several authors (see, for example, 5, 6, 7). Without exception, they define a distribution function by considering the rigidity of the Maxwell elements as distributed over a continuous range of relaxation times. Since we shall be concerned with elastico-viscous liquids, we think it more appropriate to define a distribution function of the viscosity (rather than the rigidity). The distribution function of relaxation times (or the 'relaxation spectrum') $N(\tau)$ is accordingly here defined such that $N(\tau) d\tau$ represents the total viscosity of all the Maxwell elements with relaxation times between τ and $\tau + d\tau$. The equation of state then becomes (generalizing (4))

$$p'_{ik} = 2 \int_0^{\infty} \frac{N(\tau)}{\tau} d\tau \int_{-\infty}^t e^{-(t-t')\tau} e_{ik}^{(1)}(t') dt', \quad (5)$$

and if we introduce the relaxation function Ψ , defined by

$$\Psi(t-t') = \int_0^{\infty} \frac{N(\tau)}{\tau} e^{-(t-t')\tau} d\tau, \quad (6)$$

(5) takes on the familiar form (cf. 7)

$$p'_{ik} = 2 \int_{-\infty}^t \Psi(t-t') e_{ik}^{(1)}(t') dt'. \quad (7)$$

This equation may be considered as the most general linear equation of state for an elastico-viscous liquid. Furthermore, if the relaxation spectrum $N(\tau)$ is known for all τ , the mechanical behaviour of the material is determined completely, and the spectrum can be used as a perfectly general representation of elastico-viscous liquids (cf. 7).

In slow steady motion ($e_{ik}^{(1)}$ small and independent of the time), equation (5) reduces to

$$p'_{ik} = 2e_{ik}^{(1)} \int_0^{\infty} N(\tau) d\tau, \quad (8)$$

and if we introduce a viscosity η_0 , defined as the area under the (τ, N) curve

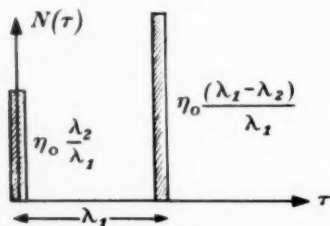
$$\eta_0 = \int_0^{\infty} N(\tau) d\tau, \quad (9)$$

equation (8) becomes

$$p'_{ik} = 2\eta_0 e_{ik}^{(1)}, \quad (10)$$

so that η_0 can be identified with the limiting viscosity at small rates of shear, as observed in steady-state experiments. This is of some importance, since the limiting viscosity η_0 is measured in many experiments and a normalizing condition (9) is immediately imposed on the relaxation spectrum. In the older representations of the spectrum as a distribution of rigidity no such simple significance can be given to the area under the curve which represents the spectrum graphically.

In terms of the above representation of the relaxation spectrum, the spectrum associated with equation (1) is given by



$$N(\tau) = \eta_0 \frac{\lambda_2}{\lambda_1} \delta(\tau) + \eta_0 \frac{(\lambda_1 - \lambda_2)}{\lambda_1} \delta(\tau - \lambda_1), \quad (11)$$

where δ is Dirac's δ -function.

3. Theoretical considerations

We can now investigate the motion when a general liquid of the type defined by equation (5) is contained between coaxial cylinders and subjected to shearing motion. This motion is supposed brought about by imposing on the outer cylinder forced harmonic angular oscillations about its axis, the inner cylinder being constrained by a torsion wire. As in (1), all tensor quantities are expressed in terms of their physical components in cylindrical polar coordinates (r, ϕ, z) , z being measured vertically upwards. Two-dimensional axially symmetric flow is assumed, with velocity components $(0, r\omega(r, t), 0)$.

The theory is very similar to that of (1), except that in this case the physical component of stress $\widehat{r\phi}'$ is given by

$$\widehat{r\phi}' = \int_{-\infty}^t \Psi(t-t') r \frac{\partial \omega(r, t')}{\partial r} dt'. \quad (12)$$

This equation has to be solved in conjunction with the equations of motion. For the simple type of motion considered in this paper the important equation of motion is that concerned with $\widehat{r\phi}'$, which may be written as

$$\frac{\partial \widehat{r\phi}'}{\partial r} + \frac{2\widehat{r\phi}'}{r} = \rho r \frac{\partial \omega}{\partial t}, \quad (13)$$

where ρ is the density.

Equations (12) and (13) have to be solved subject to the appropriate boundary conditions. A solution is possible of the form

$$\widehat{r\phi}' = \text{re } f(r) e^{2\pi i n t}, \quad \omega = \text{re } F(r) e^{2\pi i n t}, \quad (14)$$

where n is the frequency of oscillation, and $f(r)$ and $F(r)$ incorporate appropriate phase factors. Equation (12) becomes

$$f(r) = e^{-2\pi i n t} r \frac{dF}{dr} \int_{-\infty}^t \Psi(t-t') e^{2\pi i n t'} dt' = r \frac{dF}{dr} \int_0^{\infty} \Psi(\xi) e^{-2\pi i n \xi} d\xi, \quad (15)$$

and we also have, from (13),

$$\frac{d}{dr} (r^2 f) = 2\pi i n \rho r^2 F. \quad (16)$$

Eliminating F , we find

$$\left(r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} + \alpha^2 r^2 - 4 \right) f = 0, \quad (17)$$

$$\text{where } \alpha^2 = \frac{-2\pi i n \rho}{\int_0^{\infty} \Psi(\xi) e^{-2\pi i n \xi} d\xi} = \frac{-2\pi i n \rho}{\int_0^{\infty} (N(\tau) d\tau) / (1 + 2\pi i n \tau)}. \quad (18)$$

In (1), the differential equation (17) was derived with the particular expression for α^2 :

$$\alpha^2 = -\frac{2\pi i n \rho (1 + 2\pi i n \lambda_1)}{\eta_0 (1 + 2\pi i n \lambda_2)}. \quad (19)$$

This particular case can be obtained from the general expression (18) by substituting for $N(\tau)$ from (11).

The solution of (17) can be expressed in terms of Bessel functions of complex argument in the form

$$f(r) = aJ_2(\alpha r) + bY_2(\alpha r), \quad (20)$$

where a and b are constants to be determined from the boundary conditions. These are conveniently expressed in terms of the angular amplitudes of the two cylinders. If θ_1 and θ_2 are the amplitudes of angular oscillation of the inner and outer cylinders (radii r_1 and r_2 , respectively), then

$$F(r_1) = 2\pi i n \theta_1, \quad F(r_2) = 2\pi i n \theta_2 e^{ic}, \quad (21)$$

where c is a real constant. We also have the equation of motion of the inner cylinder

$$2\pi r_1^2 L f(r_1) = (K - 4\pi^2 n^2 I) \theta_1, \quad (22)$$

where I is the moment of inertia of the inner cylinder about its axis, L the length of the annular gap, and K the restoring constant of the torsion wire.

Since the measured quantity in the apparatus of Oldroyd, Strawbridge, and Toms is the ratio ϑ of the angular amplitudes of the two cylinders, the solution of equation (17), subject to the boundary conditions (21) and (22), is most conveniently expressed in the form

$$\frac{1}{\vartheta} = \frac{\theta_2}{\theta_1} = \frac{(2\pi \rho r_1^3 L) / \alpha [J_2(\alpha r_1) Y_1(\alpha r_2) - Y_2(\alpha r_1) J_1(\alpha r_2)] + (K/4\pi^2 n^2 - I) [J_1(\alpha r_1) Y_1(\alpha r_2) - Y_1(\alpha r_1) J_1(\alpha r_2)]}{(2\pi \rho r_1^2 r_2 L) / \alpha [J_2(\alpha r_1) Y_1(\alpha r_1) - Y_2(\alpha r_1) J_1(\alpha r_1)]}. \quad (23)$$

Equation (23) is unsuitable for direct computation since there appear to be no tables of the Bessel functions J_2 and Y_2 for general complex arguments. An approximate formula, valid for sufficiently large n , is obtained by substituting for the Bessel functions their asymptotic expansions in powers of $(\alpha r)^{-1}$. In this way, equation (23) reduces to the approximate formula

$$\frac{1}{\vartheta} = \left(\frac{r_1}{r_2}\right)^{\frac{3}{2}} \left[\left(1 - \frac{3(r_2 - r_1)}{8r_0^2} s\right) \cos(r_2 - r_1) \alpha + \left[\frac{15r_2 - 3r_1}{8r_0^2 \alpha} + \frac{15}{16r_0^3 \alpha^3} + \left(\alpha + \frac{3}{8r_0^2 \alpha} - \frac{45}{128r_0^4 \alpha^3}\right) s \right] \sin(r_2 - r_1) \alpha \right], \quad (24)$$

where $r_0 = \sqrt{r_1 r_2}$ and

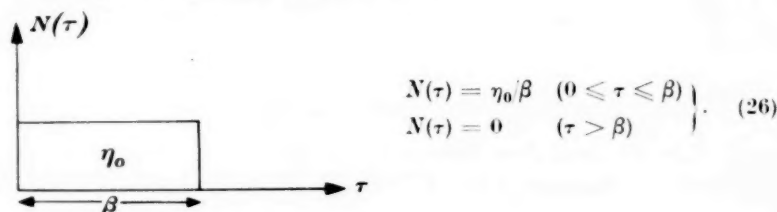
$$s = \frac{1}{2\pi \rho r_1^3 L} \left(\frac{K}{4\pi^2 n^2} - I \right). \quad (25)$$

The above equations are of the same form as those of Oldroyd (1), but here α has a different significance.

Predictions of the form of the function $\vartheta(n)$ can now be made by considering a number of simple idealized *continuous* spectra. These predictions may be compared with those for various discrete spectra of the type (11). Except where otherwise stated, the values of the apparatus constants used in the numerical calculations correspond to the experimental conditions of Oldroyd, Strawbridge, and Toms (2) designated by A (Cylinder II, Torsion wire 8).

(a) *The simple block continuous spectrum*

One of the simplest geometrical forms of the continuous relaxation spectrum corresponds to a rectangle in the (τ, N) diagram, stretching from the origin. Two constants are sufficient to specify such a spectrum, the limiting viscosity η_0 (the area under the rectangle), and the length β of the rectangle on the time scale. We have



For this simple prototype, from (18),

$$\alpha^2 = \frac{4\pi^2 n^2 \rho \beta}{\eta_0 \log(1 + 2\pi i n \beta)} \quad (27)$$

Fig. 1 contains a family of (ϑ, n) curves for liquids of this type. The curves have the same general features as the published $(\eta_0, \lambda_1, \lambda_2)$ curves (1), each showing a peak at a frequency greater than the natural frequency n_0 of free oscillation of the inner cylinder on its torsion wire, with ϑ passing through unity at a value of n almost exactly equal to n_0 .

It has not yet been possible to identify this new prototype with the behaviour of any real liquid, although it is thought that it might prove a useful approximation in the case of certain polymer solutions in a slightly higher concentration range than those considered by Oldroyd, Strawbridge, and Toms.

(b) *The displaced block spectrum*

An obvious generalization of the first prototype is obtained by displacing the rectangle along the time axis. Such a spectrum requires three constants

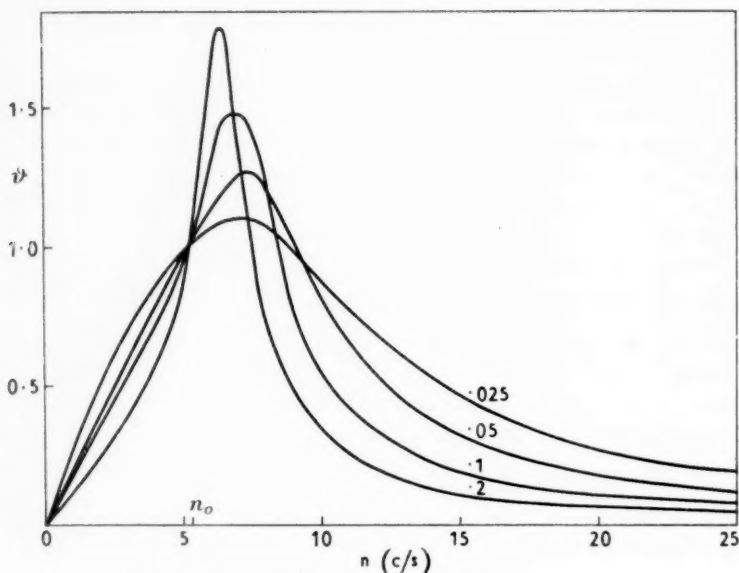


FIG. 1. The variation of $\vartheta(n)$ with β when $\rho = 1$, $\eta_0 = 8$.

η_0 , β_1 , β_2 for its description, where the block lies between β_1 and β_2 on the time scale. In this case

$$\begin{array}{c}
 \uparrow N(\tau) \\
 \begin{array}{c}
 \text{---} \beta_1 \text{---} \boxed{\eta_0} \text{---} \beta_2 \text{---} \tau
 \end{array}
 \end{array}$$

$$\left. \begin{aligned}
 N(\tau) &= 0 & (\tau < \beta_1) \\
 N(\tau) &= \eta_0/(\beta_2 - \beta_1) & (\beta_1 \leq \tau \leq \beta_2) \\
 N(\tau) &= 0 & (\tau > \beta_2)
 \end{aligned} \right\} \quad (28)$$

For general values of β_1 and β_2 ,

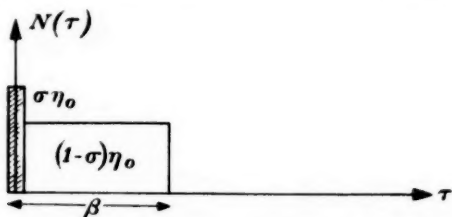
$$\alpha^2 = \frac{4\pi^2 n^2 \rho (\beta_2 - \beta_1)}{\eta_0 \log \{(1 + 2\pi i n \beta_2)/(1 + 2\pi i n \beta_1)\}} \quad (29)$$

Figs. 2 and 3 contain two families of (ϑ, n) curves for this new prototype. Fig. 2 is obtained by keeping β_2 constant and allowing β_1 to vary, while

Fig. 3 is obtained by keeping the dimensions of the block constant and displacing it along the time axis. The curves have, once again, some of the features of observed curves, but detailed comparison shows them to be inadequate to characterize the behaviour of the polymer solutions observed by Oldroyd, Strawbridge, and Toms.

(c) *The three-constant spectrum*

A more realistic prototype is one whose spectrum has a Newtonian element at the origin, with the non-Newtonian behaviour represented by a rectangular block stretching from the origin. If the Newtonian element has a viscosity $\sigma\eta_0$ ($0 \leq \sigma \leq 1$), it is convenient to characterize this new prototype by the three constants η_0 , σ , β , where β is the length of the rectangular block on the time scale. We have in this case



$$\left. \begin{aligned} N(\tau) &= \sigma\eta_0 \delta(\tau) + \frac{(1-\sigma)\eta_0}{\beta} & (0 \leq \tau \leq \beta) \\ N(\tau) &= 0 & (\tau > \beta) \end{aligned} \right\} \quad (30)$$

$$\alpha^2 = \frac{4\pi^2 n^2 \rho \beta}{[2\pi i n \beta \sigma \eta_0 + (1-\sigma)\eta_0 \log(1 + 2\pi i n \beta)]} \quad (31)$$

Fig. 4 contains a representative set of (ϑ, n) curves for three-constant idealized systems of this type.

It is clear from the graphs that these prototypes lead to (ϑ, n) curves which have the same features as the curves corresponding to the three-constant discrete spectrum (11) (1). Since Oldroyd, Strawbridge, and Toms have found those curves useful in the interpretation of experimental results on dilute polymer solutions (2, 3, 4), it is of interest to attempt an alternative interpretation in terms of the new prototype introduced in the present paper.

4. Comparison with experiment

We confine attention to the relaxation spectrum given by (30) and described by the three constants η_0 , σ , β . If this spectrum is to characterize any real liquid, η_0 has to be identified with the limiting viscosity at small

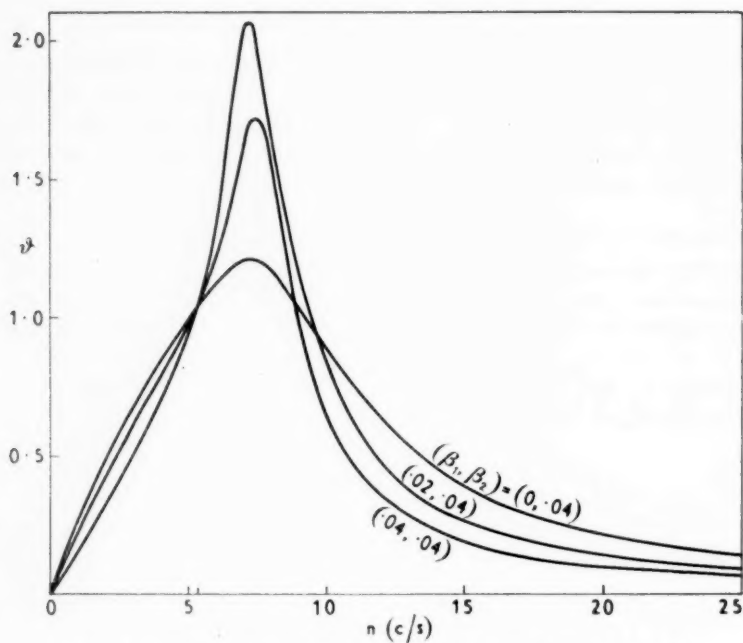


FIG. 2. The variation of $\vartheta(n)$ with β_1 and β_2 when $\rho = 1$, $\eta_0 = 8$.

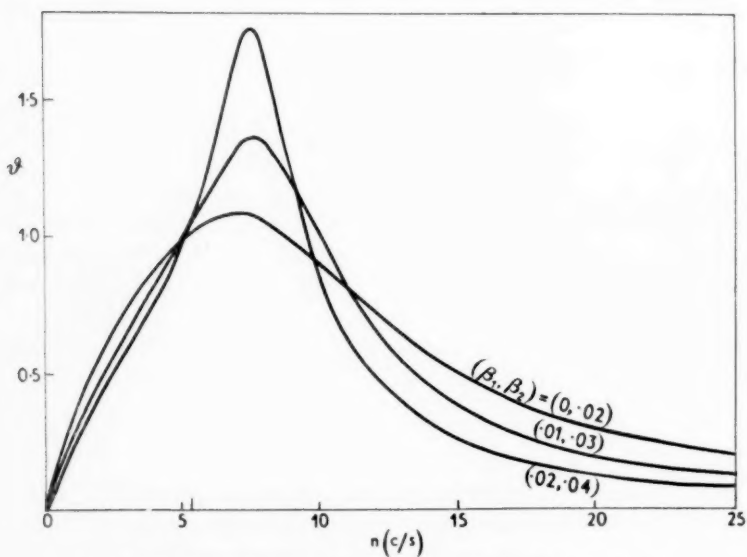


FIG. 3. The variation of $\vartheta(n)$ with β_1 and β_2 when $\rho = 1$, $\eta_0 = 8$.

rates of shear, and σ and β can then be estimated by comparing the experimental (ϑ, n) curve with families of theoretical curves based on the known ρ and η_0 and a series of values of σ and β . Figs. 5 and 6 illustrate the type of agreement which is possible when this new prototype is employed. It is clear that the experimental results, previously interpreted in terms of a discrete spectrum, can be equally well interpreted in terms of a continuous spectrum. It is, however, necessary to check that this interpretation of the

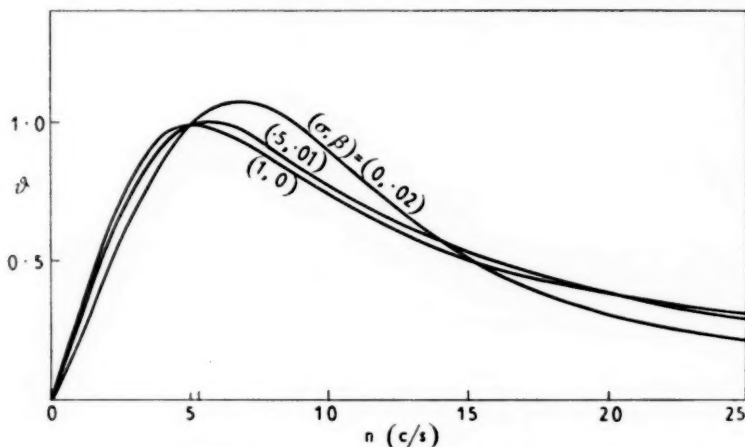


FIG. 4. The variation of $\vartheta(n)$ with σ and β when $\rho = 1$, $\eta_0 = 8$.

experimental results is independent of the particular experimental arrangement chosen and we must next compare the observations on a particular polymer solution in a number of experimental set-ups with a corresponding set of theoretical curves.

We consider for the purpose of illustration some results which have had a wide covering in the literature (1, 2, 9), obtained for a certain mixture of polymethyl methacrylate in pyridine at 25° C. This liquid, of density 0.98 g per ml, contained 30.5 g of polymer per litre and (from experiments involving steady rotation of the outer cylinder) η_0 was estimated to be 7.9 poises. In the earlier work (2), the solution was characterized by a discrete spectrum with $\lambda_1 = 0.065$ sec and $\lambda_2 = 0.015$ sec.

The values $\sigma = 0.13$, $\beta = 0.18$ sec are chosen by inspection to give the best fit for the experimental set-up A, and are used to construct a set of theoretical curves relating to five other experimental conditions. Figs. 7-12 illustrate the agreement between theory and experiment. This agreement is thought to be somewhat closer than that obtained by considering

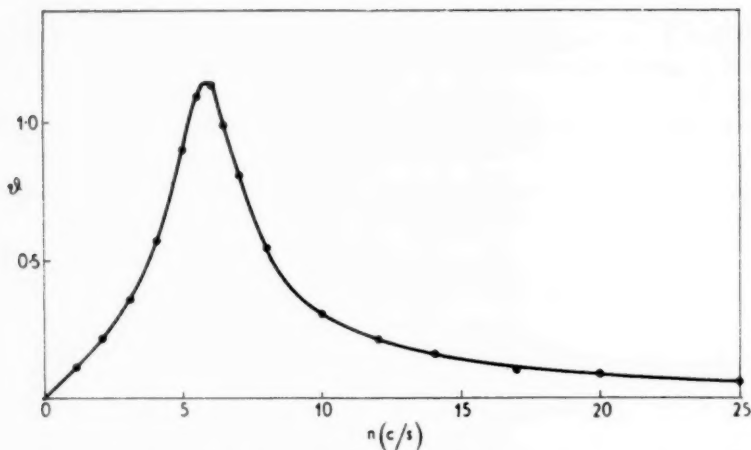


FIG. 5. The predicted relation between ϑ and n in the case of an idealized elastico-viscous liquid for which $\rho = 0.985$, $\eta_0 = 3.9$, $\sigma = 0.25$, $\beta = 0.08$; compared with experimental points for a solution of polymethyl methacrylate in pyridine (conc. 36.4 g/l) (25° C.).

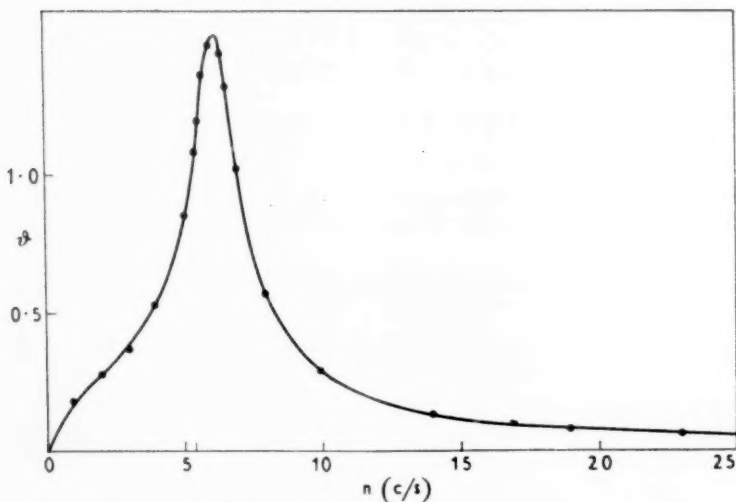


FIG. 6. The predicted relation between ϑ and n in the case of an idealized elastico-viscous liquid for which $\rho = 1$, $\eta_0 = 16$, $\sigma = 0.05$, $\beta = 0.79$; compared with experimental points for a solution of polymethyl methacrylate in pyridine (78 per cent)/water (22 per cent) (conc. = 28 g/l) (24.3° C.).

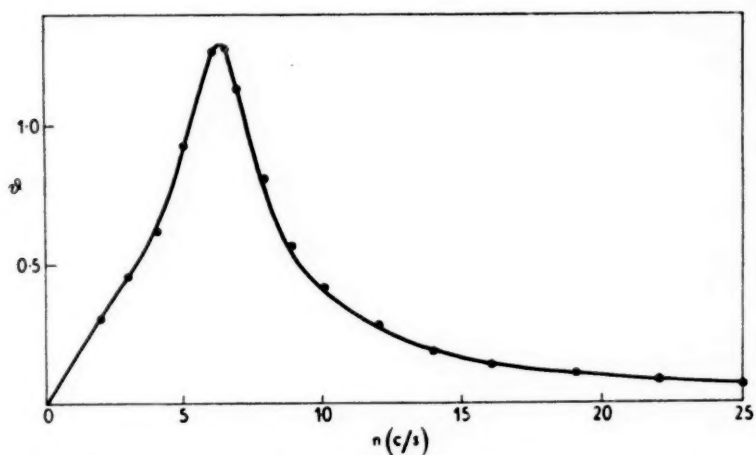


FIG. 7. Predicted (full line) and observed (\bullet) relations between ϑ and n .
(Inner Cylinder II, Torsion Wire 8.)

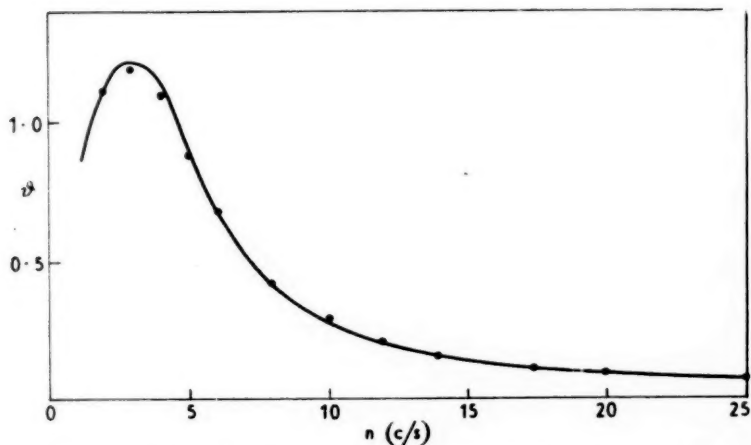


FIG. 8. Predicted (full line) and observed (\bullet) relations between ϑ and n .
(Inner Cylinder II, Torsion Wire 5 A.)

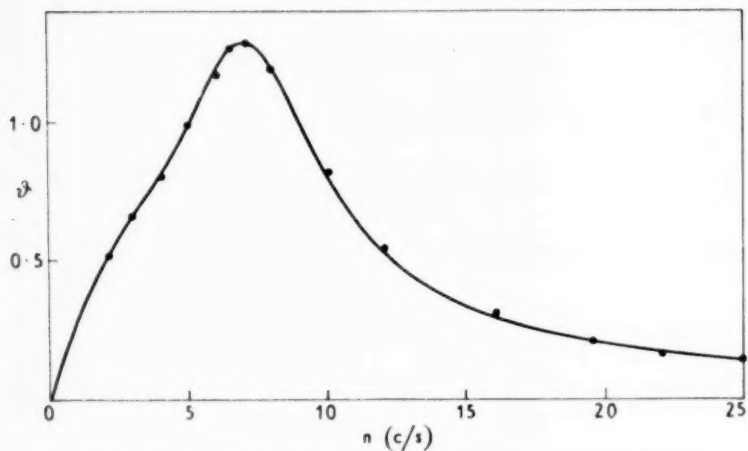


FIG. 9. Predicted (full line) and observed (●) relations between θ and n .
(Inner Cylinder I, Torsion Wire 8.)

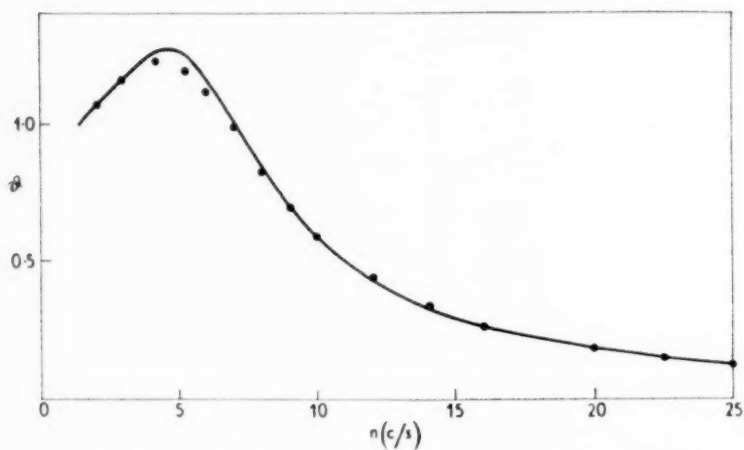


FIG. 10. Predicted (full line) and observed (●) relations between θ and n .
(Inner Cylinder I, Torsion Wire 5A.)

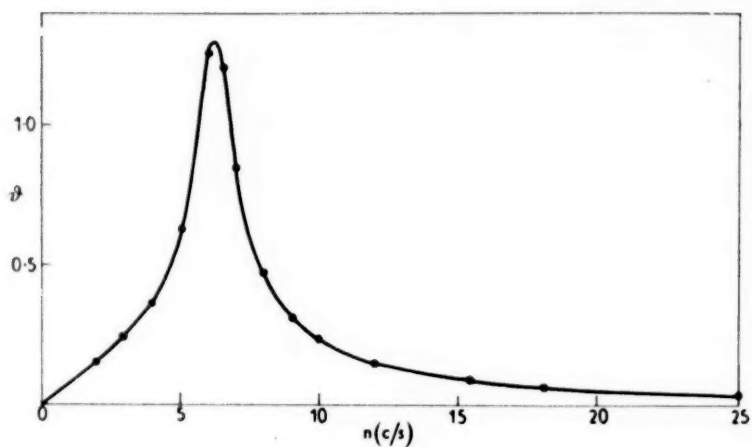


FIG. 11. Predicted (full line) and observed (●) relations between θ and n .
(Inner Cylinder III, Torsion Wire 8.)

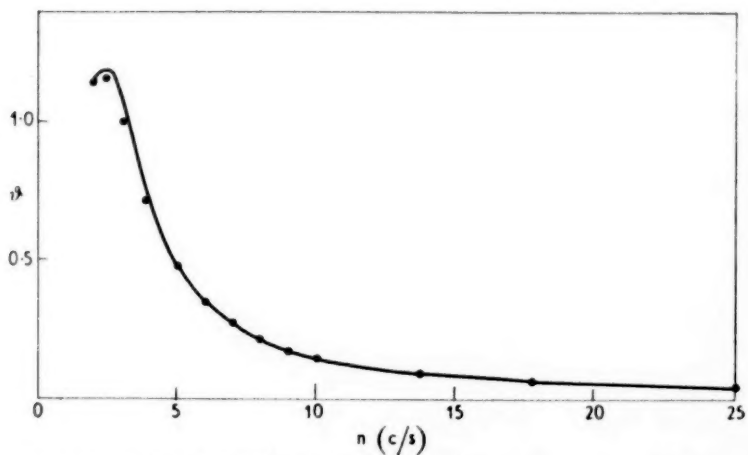


FIG. 12. Predicted (full line) and observed (●) relations between θ and n .
(Inner Cylinder III, Torsion Wire 5 A.)

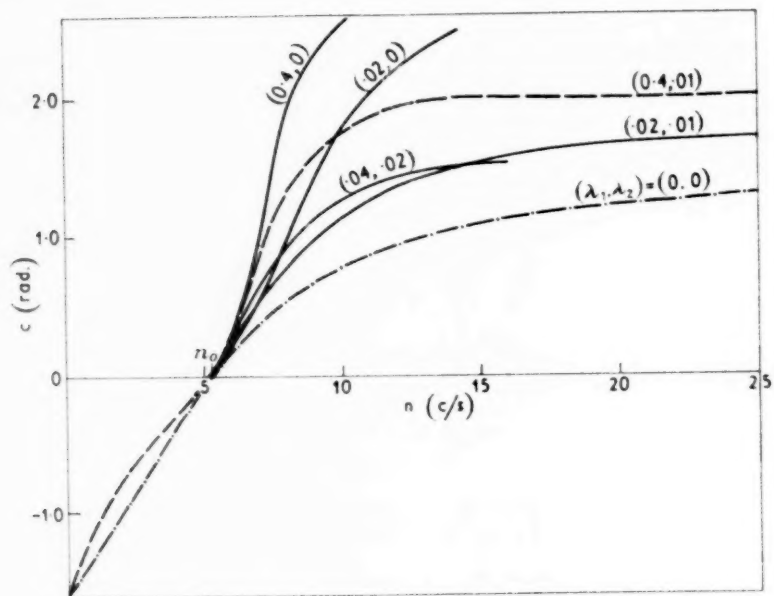


FIG. 13. The variation of $c(n)$ with λ_1 and λ_2 when $\rho = 1$, $\eta_0 = 8$.

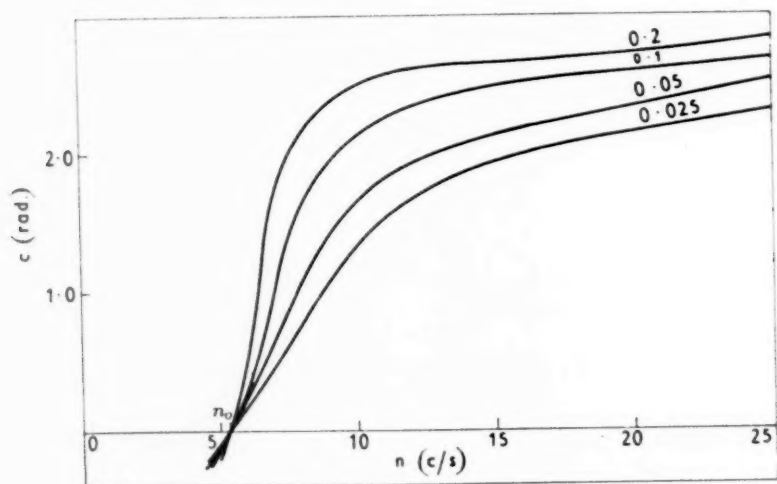


FIG. 14. The variation of $c(n)$ with β when $\rho = 1$, $\eta_0 = 8$.

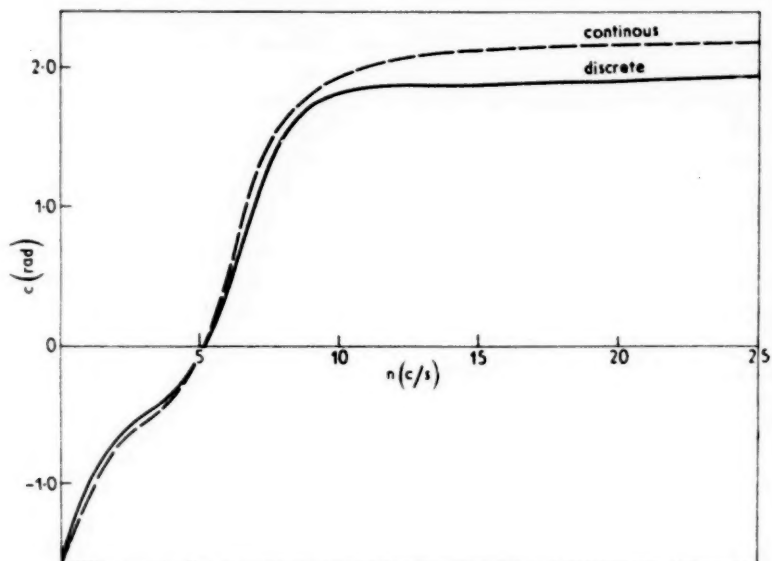


FIG. 15. Predicted relations between c and n .
(Inner Cylinder II, Torsion Wire 8.)

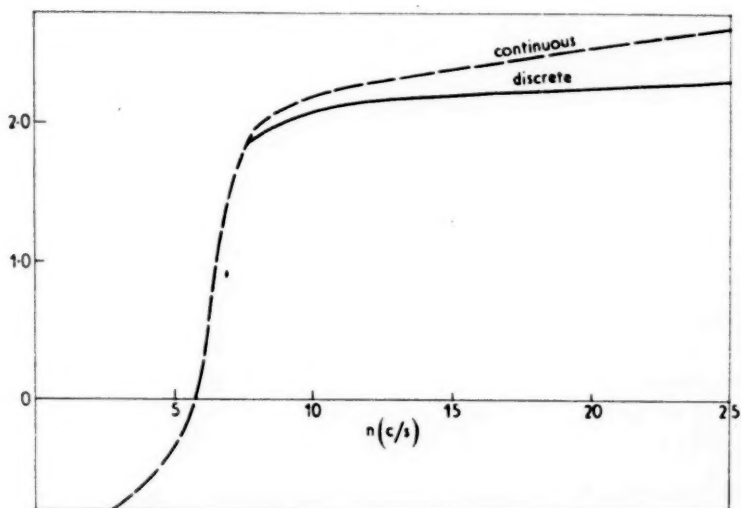


FIG. 16. Predicted relations between c and n .
(Inner Cylinder III, Torsion Wire 8.)

the $(\eta_0, \lambda_1, \lambda_2)$ model of Oldroyd (1), but the improvement is certainly not sufficient to allow one to accept the new interpretation to the exclusion of the old. Our conclusion must be that, as far as the experimental results of Oldroyd, Strawbridge, and Toms are concerned, the simple continuous spectrum defined by the three constants (η_0, σ, β) is adequate to explain the behaviour of dilute polymer solutions in a simple shearing motion at small rates of strain. This means that Oldroyd's original interpretation in terms of a three-constant model is not unique.

The findings of this section lead one to the conclusion that the experimental method of measuring amplitude ratio alone is insensitive even to certain gross changes in the form of the spectrum, and we are led to consider whether the method can be made sensitive by measuring the phase-lag between the cylinders as well.

5. The study of phase-lag

The phase-lag between the two cylinders was not measured in the experiments of Oldroyd, Strawbridge, and Toms, although it is in principle possible to measure phases and amplitudes simultaneously; and later designs of the same instrument do in fact include facilities for measuring both amplitude ratio and phase-lag (10, 11).

No new theory is required, since the phase-lag of the inner behind the outer cylinder is c in equation (21), given by

$$\frac{e^{ic}}{\vartheta} = \frac{\{(2\pi\rho r_1^3 L)/\alpha [J_2(\alpha r_1)Y_1(\alpha r_2) - Y_2(\alpha r_1)J_1(\alpha r_2)] + (K/4\pi^2 n^2 - I)[J_1(\alpha r_1)Y_1(\alpha r_2) - Y_1(\alpha r_1)J_1(\alpha r_2)]\}}{\{(2\pi\rho r_1^2 r_2 L)/\alpha [J_2(\alpha r_1)Y_1(\alpha r_1) - Y_2(\alpha r_1)J_1(\alpha r_1)]\}} \quad (32)$$

Making the same approximations as in the derivation of (24), this becomes

$$\frac{e^{ic}}{\vartheta} = \left(\frac{r_1}{r_2}\right)^3 \left\{ \left(1 - \frac{3(r_2 - r_1)}{8r_0^2} s\right) \cos(r_2 - r_1)\alpha + \left[\frac{15r_2 - 3r_1}{8r_0^2 \alpha} + \frac{15}{16r_0^3 \alpha^3} + \left(\alpha + \frac{3}{8r_0^2 \alpha} - \frac{45}{128r_0^4 \alpha^3}\right)s \right] \sin(r_2 - r_1)\alpha \right\} \quad (33)$$

Equation (33) is not valid for small values of n , and in order to determine the limiting value of c at small frequencies, we need to use the series expansions for the Bessel functions in equation (32). For very small values of n , equation (32) becomes

$$\frac{e^{ic}}{\vartheta} = \frac{-(r_2^2 - r_1^2)Ki}{8\pi^2 r_1^2 r_2^2 L \eta_0 n}, \quad (34)$$

which implies that $c \rightarrow -\frac{1}{2}\pi$ as $n \rightarrow 0$. (35)

This result is independent of the relaxation spectrum and is true for all linear elastico-viscous liquids.

Figs. 13 and 14 contain some representative theoretical (c, n) curves. Fig. 13 relates to discrete spectra of the type (11), corresponding to the particular (ϑ, n) curves contained in Fig. 1 of (1). Fig. 14 relates to simple block continuous spectra of the type (26), corresponding to the particular (ϑ, n) curves shown in Fig. 1 of the present paper. It will be noticed that c passes through zero at frequencies almost exactly equal to the natural frequency n_0 , a result which is independent of the type of relaxation spectrum considered and may be compared with the approximate coincidence of $\vartheta = 1$ and $n = n_0$ in the amplitude-ratio curves.

Figs. 15 and 16 contain, for two sets of experimental conditions, the theoretical (c, n) curves for two spectra, indistinguishable in the amplitude-ratio experiments of Oldroyd, Strawbridge, and Toms. The difference between the curves at high frequencies (as much as 20°) would certainly be detectable in phase-lag experiments, so that measurements of phase-lag could easily resolve the ambiguities arising from amplitude-ratio experiments. We conclude, therefore, that such measurements ought to be made, together with amplitude ratio, if a complete characterization of elastico-viscous liquids is required.

Acknowledgements

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UNIFORMLY STRETCHED PLATES SUBJECTED TO CONCENTRATED TRANSVERSE FORCES

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SUMMARY

This paper contains a study of the behaviour of isotropic elastic plates of various shapes subjected to uniform tension in the plane of the plate and loaded transversely by concentrated forces.

The deflexion w of the plate is governed by the partial differential equation $\Delta\Delta w - (N/D)\Delta w = 0$, where N is the tension intensity per unit length and D is the flexural rigidity of the plate. The fundamental deflexion function (Green's function for an unbounded domain) is determined and used in connexion with the method of images to construct solutions for plates of various shapes, simply supported along their boundaries.

Solutions are obtained for (a) the wedge-shaped plate with opening angle $\alpha = \pi/m$ ($m = 1, 2, 3, \dots$) and (b) the rectangular plate. It is shown that the rectangular corner plate, the infinite and semi-infinite strip, can be obtained as special cases. The rectangular corner plate is discussed in more detail.

1. The mathematical formulation of the problem

THE formulation is based on the assumption of the so-called small deflexion theory of plates. For the assumptions and derivations the reader is referred to the standard texts on the subject, e.g. (1). The deflexion normal to the surface of the plate w due to a laterally distributed load q is governed by

$$\Delta\Delta w - \kappa^2\Delta w = \frac{q}{D}, \quad (1)$$

where Δ is the Laplace operator in two independent variables and $\kappa^2 = N/D$. Here N is the tension intensity per unit length and $D = Eh^3/12(1-\mu^2)$ is the flexural rigidity of the plate.

The moments, shear forces, and the reaction distributions (along the simply supported edges parallel to the y - or x -axis) are formed as higher derivatives of w as follows:

$$\left. \begin{aligned} M_x &= -D \left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right], & M_y &= -D \left[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right] \\ M_{yx} &= -M_{xy} = -D(1-\mu) \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\}, \quad (2)$$

$$\begin{aligned}
 Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + N \frac{\partial w}{\partial x} \\
 Q_y &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x} + N \frac{\partial w}{\partial y} \\
 V_x &= -D \left[\frac{\partial^3 w}{\partial x^3} + (2-\mu) \frac{\partial^3 w}{\partial x \partial y^2} \right] + N \frac{\partial w}{\partial x} \\
 V_y &= -D \left[\frac{\partial^3 w}{\partial y^3} + (2-\mu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] + N \frac{\partial w}{\partial y}
 \end{aligned} \quad (3)$$

2. The fundamental solution

The fundamental solution of (1) is defined here as a solution axially symmetrical in nature due to the action of a concentrated force P ($q = 0$). For this case (1) reduces to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left[\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) - \kappa^2 \right] w = 0. \quad (5)$$

Its general solution is

$$w = C_1 \ln r + C_2 + C_3 I_0(\kappa r) + C_4 K_0(\kappa r), \quad (6)$$

where $I_0(\kappa r)$ and $K_0(\kappa r)$ are modified Bessel functions of order zero of first or second kind respectively.

Assuming $w = 0$ at $r = 0$ and due to the condition at $r \rightarrow \infty$ the deflexion surface reduces to the form

$$w = C [\ln r + K_0(\kappa r)]. \quad (7)$$

To determine C we consider the vertical equilibrium of a centrally loaded element separated from the rest of the plate by a cylindrical surface of radius r and obtain

$$C = -\frac{P}{2\pi N}. \quad (8)$$

Hence, the fundamental deflexion of an elastic plate subjected in its plane to hydrostatic tension and laterally to a concentrated force P is

$$w = -\frac{P}{2\pi N} [\ln r + K_0(\kappa r)], \quad (9)$$

where r is the distance from P to the investigated point of the plate. Through expansion of $K_0(\kappa r)$ it can be shown that (9) contains the characteristic singularity $(1/8\pi D)r^2 \ln r$ of a plate in bending due to a concentrated force P .

Each term on the right-hand side of (9) separately satisfies (5), since $\ln r$ is a solution of $\Delta w = 0$, and $K_0(\kappa r)$ is a solution of $(\Delta - \kappa^2)w = 0$. Thus considering (9) it can be concluded that the deflexion of a uniformly

stressed plate subjected laterally to P consists of the deflexion of a uniformly stressed membrane subjected to P , $w = -(P/2\pi N)\ln r$, and the solution of $(\Delta - \kappa^2)w = 0$ which allows for the 'flexural rigidity' of the plate. It is in fact the \ln -term of the flexural rigidity expression which makes w finite at $r = 0$.

3. Solution of boundary value problems

For plates bounded by simply supported straight boundaries solutions can be obtained using the 'method of images' (1). A more general method suitable for plates of arbitrary shape and arbitrary degree of fixity is based on the fact that the deflexion can be represented in the form $w = w_f + w_b$, where w_f is the fundamental deflexion and w_b are functions which satisfy $\Delta w_b - \kappa^2 w_b = 0$, adjusting the w_f values at the boundaries to the prescribed ones. The exactness of w depends then mainly on how accurately w_b can correct w_f at the boundary to the prescribed values. It should be noted that the method of images is a special case of the above.

In the following, plates of various shapes will be treated by the method of images.

4. Wedge-shaped plates. The rectangular corner plate

Solutions for wedge-shaped plates can be obtained in closed form by arranging an even number of forces P on a circle about the tip of a plate. For a plate with an opening angle $\alpha = \pi/m$ ($m = 1, 2, \dots$) subjected to a force P in an arbitrary point inside the plate

$$w = -\frac{P}{2\pi N} \sum_{n=1}^{2m} (-1)^{n+1} [\ln r_n + K_0(\kappa r)], \quad (10)$$

where the r_n 's are the distances of the forces P to the point of the plate under investigation. For the rectangular corner plate $m = 2$ (Fig. 1), (10) reduces to

$$w = -\frac{P}{2\pi N} \left[\ln \frac{r_1 r_3}{r_2 r_4} + K_0(\kappa r_1) - K_0(\kappa r_2) + K_0(\kappa r_3) - K_0(\kappa r_4) \right], \quad (11)$$

where

$$\begin{aligned} r_1 &= \sqrt{(x-x_0)^2 + (y-y_0)^2}, & r_2 &= \sqrt{(x-x_0)^2 + (y+y_0)^2}, \\ r_3 &= \sqrt{(x+x_0)^2 + (y+y_0)^2}, & r_4 &= \sqrt{(x+x_0)^2 + (y-y_0)^2}. \end{aligned} \quad (12)$$

It can be shown that the boundary conditions $w = 0$ and $\Delta w = 0$ along the supported edges are satisfied.

To find the influence of the ratio $\kappa^2 = N/D$ upon the deflexions, (11) is reduced for the case when P acts at a point on the bisector radius (at $r = r_0$) to an equation of the deflexion curve along that line. The numerical evaluation of this equation for different values κ^2 is shown in Fig. 2.

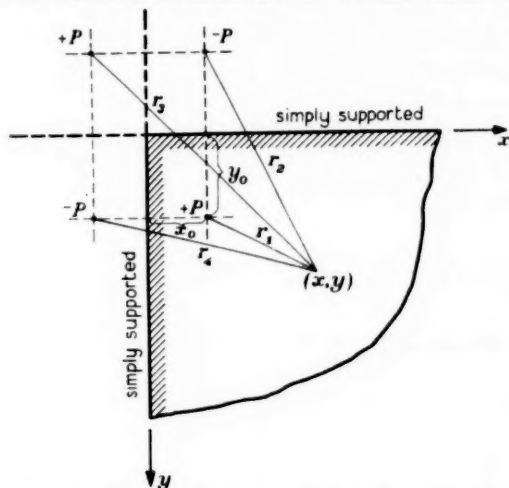
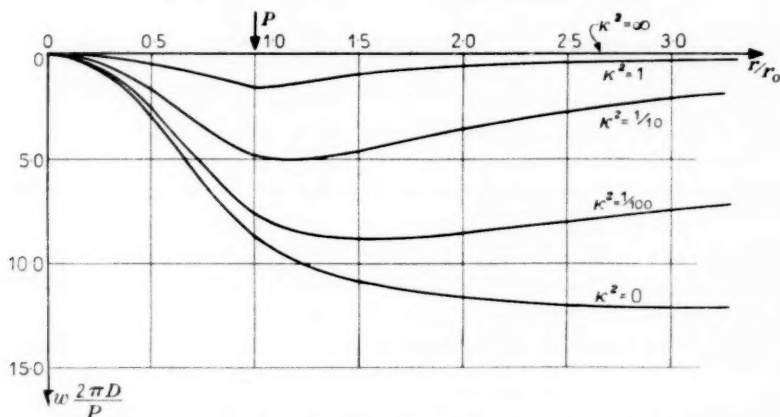


FIG. 1. Force system for the rectangular corner plate.

FIG. 2. Corner plate. Deflection curves along the bisector radius for different values of κ^2 . $r_0 = 5$.

It is interesting to note that whereas for the case $N \neq 0$ the deflections at infinity tend to zero, for the particular case $N = 0$ the deflections at infinity are infinite.

Because of the assumptions made in deriving (1) a negative concentrated corner reaction appears in the analytical solution in addition to the reactions distributed along the supports:

$$R = 2D(1-\mu) \left[\frac{\partial^2 w}{\partial x \partial y} \right]_{x=x_1, y=y_1} \quad (13)$$

h h

Substituting (11) into (13) and setting $x = 0$ and $y = 0$ we obtain

$$R = \frac{4(1-\mu)P}{\pi} \left\{ \frac{2x_0 y_0}{\kappa^2(x_0^2 + y_0^2)^2} - \frac{x_0 y_0}{(x_0^2 + y_0^2)} K_2(\kappa \sqrt{x_0^2 + y_0^2}) \right\}. \quad (14)$$

This is the corner reaction for an arbitrary position of P inside the plate. The numerical evaluation of R as function of κ^2 for the case $x_0 = y_0 = a$ is shown in Fig. 3.

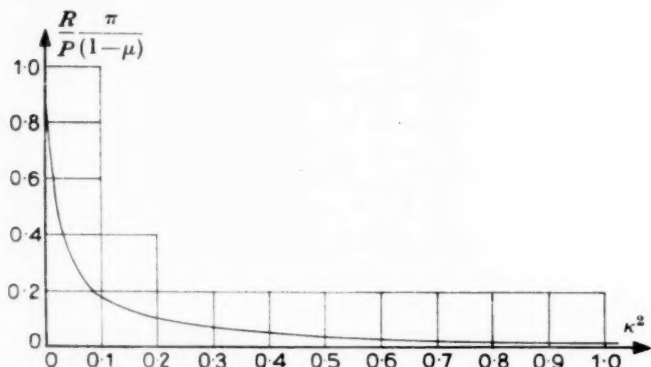


Fig. 3. Corner plate. Corner reaction R as function of κ^2 .
 $\mu = 0.3, a = 10$.

The distributed reactions along the supports are obtained substituting (11) into (4). For example the reactions at the support along the y -axis are

$$[V_x]_{x=0} = \frac{P}{\pi} \left\{ \frac{2(1-\mu)}{\kappa^2} \left[-\frac{x_0[3(y-y_0)^2 - x_0^2]}{[x_0^2 + (y-y_0)^2]^3} + \frac{x_0[3(y+y_0)^2 - x_0^2]}{[x_0^2 + (y+y_0)^2]^3} \right] - \right. \\ \left. - \frac{x_0}{x_0^2 + (y-y_0)^2} [(1-\mu)K_2(\kappa \sqrt{x_0^2 + (y-y_0)^2}) - 1] + \right. \\ \left. + \frac{x_0}{x_0^2 + (y+y_0)^2} [(1-\mu)K_2(\kappa \sqrt{x_0^2 + (y+y_0)^2}) - 1] + \right. \\ \left. + (1-\mu)\kappa \left[\frac{x_0(y-y_0)^2}{[x_0^2 + (y-y_0)^2]^3} K_3(\kappa \sqrt{x_0^2 + (y-y_0)^2}) - \right. \right. \\ \left. \left. - \frac{x_0(y+y_0)^2}{[x_0^2 + (y+y_0)^2]^3} K_3(\kappa \sqrt{x_0^2 + (y+y_0)^2}) \right] \right\}. \quad (15)$$

To illustrate the influence of the stretching forces upon the reaction distributions, graphs of $[V_x]_{x=0}$ for $\mu = 0.3$ and different values of κ^2 are shown in Fig. 4, for the case $x_0 = y_0 = a$. We find that the reactions for various values of κ^2 are bracketed between two limiting cases, namely, that of $\kappa^2 = 0$ (plate in bending) and the one corresponding to $\kappa^2 = \infty$ (stretched membrane). It is to be noted that the maximum value of $[V_x]_{x=0}$ does not

occur at the point on the boundary closest to P , not even in the case of stretched membrane. It can also be observed that the areas enclosed by the reaction distribution for different κ^2 values (but for the same value of the force P) are not equal. This is due to the fact that the corner reaction R varies with changing κ^2 and because of the vertical equilibrium, for decreasing R the area enclosed by V has to decrease since

$$R + P - 2 \int_0^{\infty} [V_x]_{x=0} dy = 0. \quad (16)$$

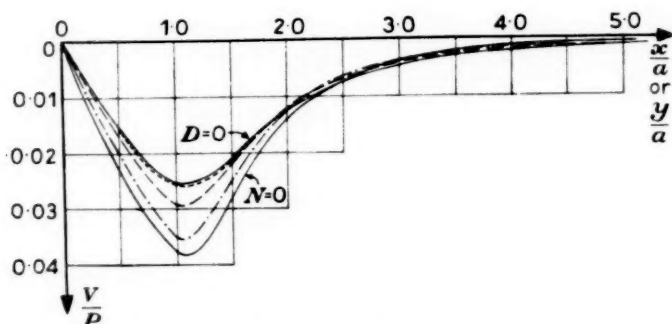


FIG. 4. Corner plate. Reaction distribution along $y = 0$ or $x = 0$.
 — $\kappa^2 = \infty$ or 0 , --- $\kappa^2 = 1$, - - - $\kappa^2 = \frac{1}{10}$, — — — $\kappa^2 = \frac{1}{100}$, $\mu = 0.3$, $a = 10$.

5. The rectangular plate

A system of forces as shown in Fig. 5 periodically repeating in the positive and negative directions of the x - and y -axis produces a series of deflexion surfaces each of them equivalent to that of a rectangular plate simply supported along its boundaries and subjected at an arbitrary point (x_0, y_0) to a force P .

The deflexion surface is

$$w = \frac{P}{2\pi N} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left\{ \ln \left(\frac{r'_{nm} \rho_{nm}}{r_{nm} \rho'_{nm}} \right) - K_0(\kappa r_{nm}) + K_0(\kappa r'_{nm}) + K_0(\kappa \rho_{nm}) - K_0(\kappa \rho'_{nm}) \right\}, \quad (17)$$

where

$$\left. \begin{aligned} r_{nm} &= \sqrt{\{(x-2na-x_0)^2 + (y-2mb-y_0)^2\}} \\ \rho_{nm} &= \sqrt{\{(x-2na-x_0)^2 + (y-2mb+y_0)^2\}} \\ r'_{nm} &= \sqrt{\{(x-2na+x_0)^2 + (y-2mb-y_0)^2\}} \\ \rho'_{nm} &= \sqrt{\{(x-2na+x_0)^2 + (y-2mb+y_0)^2\}} \end{aligned} \right\}. \quad (18)$$

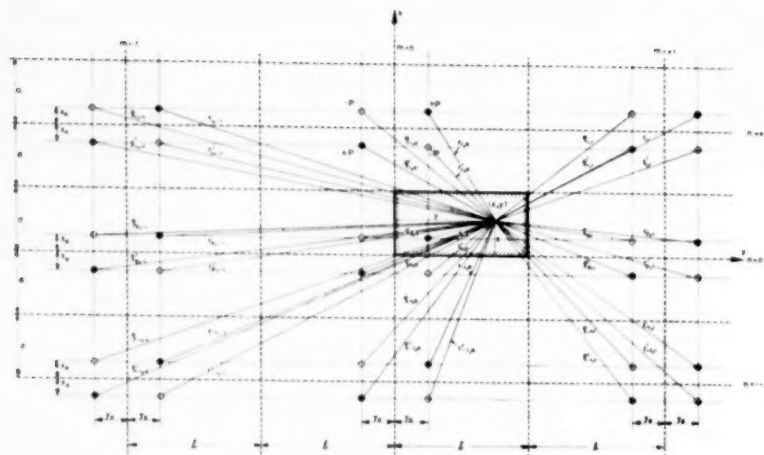


FIG. 5. Force system for the rectangular plate.

The bending moments are derived by substituting (17) into the first two relations of (2).

$$\begin{aligned}
 M_x = \frac{P}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} & \left\{ \frac{(1-\mu)}{\kappa^2} \left[-\frac{(x-x_0-2na)^2 - (y-y_0-2mb)^2}{r_{nm}^4} + \right. \right. \\
 & + \frac{(x+x_0-2na)^2 - (y-y_0-2mb)^2}{r'_{nm}{}^4} - \frac{(x+x_0-2na)^2 - (y+y_0-2mb)^2}{\rho_{nm}^4} + \\
 & + \left. \frac{(x-x_0-2na)^2 - (y+y_0-2mb)^2}{\rho_{nm}^4} \right] - \frac{(1+\mu)}{\kappa} \left[\frac{1}{r_{nm}} K_1(\kappa r_{nm}) - \right. \\
 & - \frac{1}{r'_{nm}} K_1(\kappa r'_{nm}) - \frac{1}{\rho_{nm}} K_1(\kappa \rho_{nm}) + \left. \frac{1}{\rho'_{nm}} K_1(\kappa \rho'_{nm}) \right] + \\
 & + \left[\frac{(x-x_0-2na)^2 + \mu(y-y_0-2mb)^2}{r_{nm}^2} K_2(\kappa r_{nm}) - \right. \\
 & - \left[\frac{(x+x_0-2na)^2 + \mu(y-y_0-2mb)^2}{r'_{nm}{}^2} K_2(\kappa r'_{nm}) - \right. \\
 & - \left. \left[\frac{(x-x_0-2na)^2 + \mu(y+y_0-2mb)^2}{\rho_{nm}^2} K_2(\kappa \rho_{nm}) + \right. \right. \\
 & + \left. \left. \frac{(x+x_0-2na)^2 + \mu(y+y_0-2mb)^2}{\rho_{nm}^2} K_2(\kappa \rho'_{nm}) \right] \right\}
 \end{aligned} \quad (19)$$

$$M_y = \frac{P}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left\{ -\frac{(1-\mu)}{\kappa^2} \left[-\frac{(x-x_0-2na)^2 - (y-y_0-2mb)^2}{r_{nm}^4} + \right. \right. \\ \left. \left. + \frac{(x+x_0-2na)^2 - (y-y_0-2mb)^2}{r_{nm}^4} - \frac{(x+x_0-2na)^2 - (y+y_0-2mb)^2}{\rho_{nm}^4} + \right. \right. \\ \left. \left. + \frac{(x-x_0-2na)^2 - (y+y_0-2mb)^2}{\rho_{nm}^4} \right] - \frac{(1+\mu)}{\kappa} \left[\frac{1}{r_{nm}} K_1(\kappa r_{nm}) - \right. \right. \\ \left. \left. - \frac{1}{r'_{nm}} K_1(\kappa r'_{nm}) - \frac{1}{\rho_{nm}} K_1(\kappa \rho_{nm}) + \frac{1}{\rho'_{nm}} K_1(\kappa \rho'_{nm}) \right] + \right. \\ \left. + \frac{[\mu(x-x_0-2na)^2 + (y-y_0-2mb)^2]}{r_{nm}^2} K_2(\kappa r_{nm}) - \right. \\ \left. - \frac{[\mu(x+x_0-2na)^2 + (y-y_0-2mb)^2]}{r_{nm}^2} K_2(\kappa r'_{nm}) - \right. \\ \left. - \frac{[\mu(x-x_0-2na)^2 + (y+y_0-2mb)^2]}{\rho_{nm}^2} K_2(\kappa \rho_{nm}) + \right. \\ \left. + \frac{[\mu(x+x_0-2na)^2 + (y+y_0-2mb)^2]}{\rho_{nm}^2} K_2(\kappa \rho'_{nm}) \right\} \quad (20)$$

and

$$\frac{M_x + M_y}{1+\mu} = -D\Delta w \\ = -\frac{P}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \{ -K_0(\kappa r_{nm}) + K_0(\kappa r'_{nm}) + K_0(\kappa \rho_{nm}) - K_0(\kappa \rho'_{nm}) \}. \quad (21)$$

The sufficient condition for validity of term-by-term differentiation of an infinite series in two variables is satisfied since (17) as well as its first, second, and third derivatives with respect to x and y are continuous functions of x and y in the whole domain (except the second and third derivatives at $x = y = 0$ where they have a singularity).

For solutions expressed in terms of infinite series, the problem of convergence, as well as the rate of convergence, is of considerable importance. From the theory of infinite series it is known (2) that if the terms of a series

$$u_1 - u_2 + u_3 - u_4 + \dots \quad (22)$$

be of alternate sign, it is necessary and sufficient for the convergence of the series that

$$u_n \geq u_{n+1} \text{ for every value of } n, \text{ and } \lim_{n \rightarrow \infty} u_n = 0. \quad (23)$$

Because $\sum P = 0$ the solution obtained using the method of images forms (for each point on the plate) an alternating series. It can be shown

that the condition in (23) is satisfied and that the resulting series are convergent. The validity of condition (23) for the cases treated here can also be proved by considering the physical picture. Namely, if a stretched plate is subjected to a concentrated force P , the effects (like w , M , V) are localized and will decrease with increasing distance from P . As the terms of increasing n represent the effects of forces P acting at increasing distances from the point under consideration, these terms will decrease with increasing

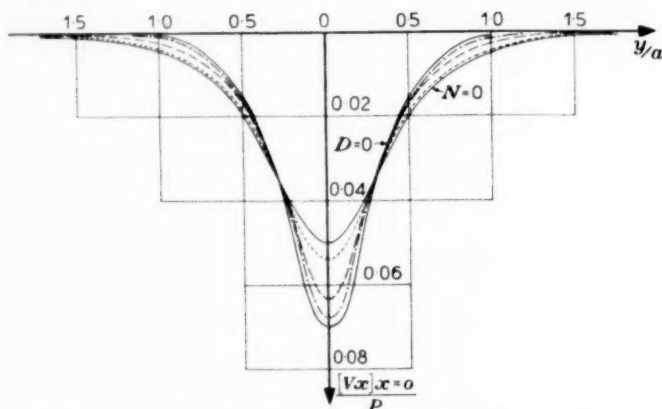


FIG. 6. Infinite strip. Reaction distributions along $x = 0$ or $x = a$ for $\mu = 0.25$.
 — $\kappa^2 = \infty$ or 0 , $\kappa^2 = 1$, — — — $\kappa^2 = \frac{1}{10}$, — · — $\kappa^2 = \frac{1}{100}$, $a = 10$.

n , and this proves physically that condition (23) is satisfied for the cases under consideration.

In representing the deflexion surface due to P by means of a Fourier series, as done in (1), 310, we deal with a force system similar to the one represented in Fig. 5. In the case of the sine series, with n increasing, each additional term improves the approximate representation of the 'concentrated' forces, all of which are already acting on the infinite plate. The double sine series satisfies automatically the condition of simply supported boundaries. In the case of a representation of the deflexion surface by means of fundamental deflexions, (17), each additional term represents an additional 'complete' force P applied on the infinite plate, which contributes towards the formation of simply supported boundaries.

From what has been said above, one can anticipate the very slow convergence of the formulae for the bending moments, expressed as double sine series, in the neighbourhood of P , since it is rather 'inconvenient' for a sine or cosine series to represent these strong irregularities. There will be a quicker convergence for moments derived by means of the fundamental

solutions, since in the vicinity of (x_0, y_0) the main part of the moments is represented by the fundamental solution of P at (x_0, y_0) , and the contribution of moments caused by the other forces P is diminishing rapidly with increasing distance from (x_0, y_0) .

It is to be noted that for $m = 0$ and $n = 0$ the expressions obtained for the rectangular plate reduce to that of a rectangular corner plate. The summation of the terms $\rho_{n,0}$ and $r_{n,0}$ for $n = 1, 2, \dots$ only, yields the solution for a semi-infinite strip. Summing the $r_{n,0}$'s for $n = 1, 2, \dots$ only, we obtain the solution for an infinite strip. In Fig. 6 the distribution of reactions is shown for the infinite strip of width a , subjected to a force P at $(0, \frac{1}{2}a)$. Also, here it can be seen that the reactions for various values of κ^2 are enclosed between the two limiting cases: that of a stretched membrane and the one of a plate in bending.

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ON THE PROPAGATION OF SOUND IN A LAYERED FLUID MEDIUM

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[Received 8 October 1959]

SUMMARY

In this paper the Green's function is derived for a layered fluid of finite depth subject to a general boundary condition at the bottom. The limiting form of the Green's function as the depth is increased indefinitely is considered, and a physical interpretation of the results given.

The relation between the limiting form of our problem and the same problem set with a radiation condition at ∞ is examined. A particular example is worked in detail.

The form of the solution due to a transient pulse is noted; also some aspects of the problem of numerical solution. A general formula for the group velocity is derived.

1. Introduction

THE basic problem to be discussed here is the solution of the wave equation in a medium in which the velocity of propagation c depends on the depth coordinate z alone. In these circumstances the wave equation can be written

$$\nabla^2 P = \frac{1}{\{c(z)\}^2} \frac{\partial^2 P}{\partial t^2} \quad (1)$$

or, in the slightly more general form,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \rho \frac{\partial}{\partial z} \left\{ \frac{1}{\rho} \frac{\partial P}{\partial z} \right\} = \frac{1}{\{c(z)\}^2} \frac{\partial^2 P}{\partial t^2}$$

which allows for changes in density ρ with depth. The latter equation is amenable to the same treatment as (1) though we shall have little cause to use it. An exception occurs when ρ is allowed to have step discontinuities corresponding to layers of immiscible fluid of different densities as in section 5.

We consider first solutions which have a simple harmonic dependence on time. This leads us to consider the equation (Helmholtz equation)

$$\nabla^2 P + \frac{\omega^2}{\{c(z)\}^2} P = 0. \quad (1a)$$

Together with this equation we consider the boundary conditions:

(a) radially outgoing wave (or radiation condition) as $r = \sqrt{(x^2 + y^2)} \rightarrow \infty$.

For our purposes it will be satisfactory to say that $f(v)$ satisfies a radiation condition at ∞ if

$$f(v) \rightarrow Ae^{-iv} + g(v), \quad \text{and} \quad g(v) \rightarrow 0, \quad v \rightarrow \infty.$$

This condition is a particular case (with $g(v)$ suitably restricted) of the Sommerfeld condition

$$\lim_{v \rightarrow \infty} v^{\frac{1}{2}} \left| \frac{df}{dv} + if \right| = 0$$

which has been employed as a sufficient condition in proving uniqueness theorems for infinite regions (see for example, Stoker (1));

(b) $P_{(z=0)} = 0$, pressure release surface; and

$$(c) \quad \cos \alpha P_{(z=H)} + \sin \alpha \frac{\partial P_{(z=H)}}{\partial z} = 0.$$

Condition (c) contains $P(H) = 0$ (pressure release bottom), and $\partial P(H)/\partial z = 0$ (rigid bottom) as special cases so that it is physically appropriate. To understand its mathematical significance we introduce cylindrical coordinates in equation (1a) and then separate the variables.

By putting $P = \theta(r)\phi(z)$ we obtain the pair of equations

$$\nabla^2 \theta + k^2 \theta = 0 \quad (2)$$

and

$$\frac{d^2 \phi}{dz^2} + \left(\frac{\omega^2}{c(z)^2} - k^2 \right) \phi = 0, \quad (3)$$

where k^2 is the separation constant.

Thus we see that (c) is the most general condition at $z = H$ which depends solely on the values of ϕ and $d\phi/dz$ at $z = H$, and which makes equation (3) self-adjoint on $(0, H)$. Thus equation (3) has a discrete sequence of real eigenvalues $k_n^2 \rightarrow -\infty$ and eigenfunctions ϕ_n as its possible solutions. Boundary condition (a) implies that we must choose the Bessel function $H_0^{(2)}$ as the solution of (2).

The choice of $\bar{k} = -k^2$ as the separation constant leads to a more conventional form of the eigenvalue problem [equation (3)], and to the modified Bessel function of the second kind $K_0(\bar{k}r)$ as the solution of (2)—again perhaps the more conventional choice. We have adopted our notation to conform with that used by Pekeris in his paper (2) in order to be able to make direct comparisons with his work.

2. Construction of the Green's function

To find the Green's function for (1a), that is the solution due to a point source, we assume that it can be expressed in terms of the Green's

functions of (2) and (3); this suggests that its form is

$$G(r; z, z_P) = -\frac{1}{4}i \sum_{n=0}^{\infty} H_0^{(2)}(k_n r) \frac{\phi_n(z)\phi_n(z_P)}{f(k_n^2)},$$

where we have to determine $f(k_n^2)$, and where we have written

$$r = \sqrt{(x-x_P)^2 + (y-y_P)^2}.$$

We will assume in this section that the ϕ_n are normalized so that

$$\int_0^H \phi_n^2 dz = 1.$$

Now $G(r; z, z_P)$ has the defining property that, if

$$\nabla^2 \Phi + [\omega^2 \{c(z)\}^2] \Phi = R(\bar{r})Z(z),$$

then

$$\Phi(P) = 2\pi \iint G(r; z, z_P) R(\bar{r})Z(z) \bar{r} d\bar{r} dz,$$

where P denotes the point with coordinates (x_P, y_P, z_P) and $\bar{r} = \sqrt{(x^2 + y^2)}$.

Substituting for $G(r; z, z_P)$ we get

$$\begin{aligned} \Phi(P) &= \sum_n -\frac{1}{2}\pi i \left\{ \int H_0^{(2)}(k_n r) R(\bar{r}) \bar{r} d\bar{r} \right\} \left\{ \int \frac{\phi_n(z)\phi_n(z_P)}{f(k_n^2)} Z(z) dz \right\} \\ &= \sum_n \Phi_n(P). \end{aligned}$$

Now we have

$$\nabla_{r_P}^2 \Phi_n(P) = -k_n^2 \Phi_n(P) + R(r_P) \int \frac{\phi_n(z)\phi_n(z_P)}{f(k_n^2)} Z(z) dz$$

since $-\frac{1}{4}iH_0^{(2)}(k_n r)$ is the Green's function of (2), and

$$\frac{\partial^2 \Phi_n(P)}{\partial z^2} + \frac{\omega^2}{\{c(z_P)\}^2} \phi_n(P) = k_n^2 \Phi_n(P),$$

since $\phi_n(z_P)$ satisfies (3). That is,

$$\begin{aligned} \nabla^2 \Phi(P) + \frac{\omega^2}{\{c(z_P)\}^2} \Phi(P) &= R(r_P) \int Z(z) \left\{ \sum_n \frac{\phi_n(z)\phi_n(z_P)}{f(k_n^2)} \right\} dz \\ &= R(r_P) Z(z_P). \end{aligned}$$

Hence

$$\sum \frac{\phi_n(z)\phi_n(z_P)}{f(k_n^2)} = \delta(z - z_P),$$

which implies that $f(k_n^2) \equiv 1$. Thus we have

$$G(r; z, z_P) = -\frac{1}{4}i \sum_{n=0}^{\infty} H_0^{(2)}(k_n r) \phi_n(z)\phi_n(z_P). \quad (4)$$

This series converges for all values of its argument which lie within the volume considered provided that we interpret $\sqrt{(-k^2)}$ as $-i|k|$. The Bessel function $H_0^{(2)}$ then decays exponentially for $r > 0$ as $k_n^2 \rightarrow -\infty$.

We here introduce the term 'mode' to describe the individual terms in (4).

Those modes with $k_n^2 < 0$ are sometimes called 'evanescent' to distinguish them from the propagated modes ($k_n^2 > 0$).

The argument we have used above can be made rigorous for the sequence of integrals I_k defined by

$$I_k = \int Z(z) \left\{ \sum_{n=0}^k \phi_n(z) \phi_n(z_P) \right\} dz.$$

It defines the generalized function $\delta(z-z_P)$ provided $Z(z)$ is suitably restricted (see, for instance, Lighthill (3)).

3. Spectral analysis of equation (3)

In this section we consider the limiting form of the Green's function given in equation (4) as the depth is increased without limit ($H \rightarrow \infty$).

We assume that $c(z) \rightarrow$ positive limit $\leq \infty$. Therefore $\omega^2/\{c(z)\}^2 \rightarrow$ limit $a^2 \geq 0$. For convenience we write equation (3) as

$$\frac{d^2\phi}{dz^2} + \left\{ \frac{\omega^2}{\{c(z)\}^2} - a^2 + a^2 - k^2 \right\} \phi = 0,$$

$$\text{or} \quad \frac{d^2\phi}{dz^2} + \{q(z) + \lambda\} \phi = 0. \quad (3^*)$$

Provided that $\int_0^\infty |q(z)| dz < \infty$, then (3*) has a fundamental set of solutions

$$M(z) \rightarrow e^{i\sqrt{\lambda}z} \quad (z \rightarrow \infty),$$

$$\text{and} \quad N(z) \rightarrow e^{-i\sqrt{\lambda}z} \quad (z \rightarrow \infty)$$

(these being the leading terms of asymptotic expansions valid in a neighbourhood of $z = \infty$). In particular $N(z)$ satisfies a radiation condition at ∞ . If $|q(z)|$ is not integrable on $(0, \infty)$ then we find asymptotic expansions which have a more complicated form but which behave similarly for large values of z . To simplify our arguments we will assume that $|q(z)|$ is integrable.

The characteristic equation for equation (3) (the equation whose zeros as a function of k^2 give the eigenvalues) is

$$\begin{vmatrix} M(0) & N(0) \\ \cos \alpha M(H) + \sin \alpha \frac{dM(H)}{dz} & \cos \alpha N(H) + \sin \alpha \frac{dN(H)}{dz} \end{vmatrix} = 0.$$

We now show that, if $-q_{\max} < \lambda < 0$, then $\lambda(H)$ takes discrete values which tend to definite limits independent of α as $H \rightarrow \infty$. This follows because

(a) for finite H the problem is self-adjoint with, therefore, a discrete spectrum, and hence only a finite number of eigenvalues in this range;

(b) M and N are integral functions of λ for finite (fixed) values of z , see (4), chapters 1 and 7;

(c) we have

$$M(z) \rightarrow \infty \quad (z \rightarrow \infty),$$

and

$$N(z) \rightarrow 0 \quad (z \rightarrow \infty),$$

for λ in the stated range. Hence the characteristic equation for large H can be written

$$N(0) = \frac{M(0)[\cos \alpha N(H) + \sin \alpha \{dN(H)/dz\}_1]}{\cos \alpha M(H) + \sin \alpha \{dM(H)/dz\}_1},$$

and the magnitude of the right-hand side can be made arbitrarily small by taking H large enough. Thus, as $H \rightarrow \infty$, the roots of this equation tend to the roots of $N(0) = 0$; and, as $N(0)$ is an integral function of λ , the roots of $N(0) = 0$ cannot have a finite limit point. By a similar argument we can show that the eigenfunctions also tend to definite limits, and that the difference between any eigenfunction and its limit function can be made arbitrarily small on $(0, H)$ by taking H large enough.

In limit $H \rightarrow \infty$ values of $\lambda \geq 0$ lie in the continuous spectrum of (3). To show this we have to prove that the number of eigenvalues between any two fixed values of λ increases without limit as $H \rightarrow \infty$.

That this must be so follows from the oscillation properties of the eigenfunctions—the n th eigenfunction being characterized by having exactly n zeros in the interior of $(0, H)$. From the asymptotic forms for M and N we see that these oscillation properties must be similar to those of $\sin \sqrt{\lambda} z$ for $\lambda > 0$ and z large enough. The change in the number of zeros of $\sin \sqrt{\lambda} z$ on $(0, H)$ within a given range of λ tends to ∞ as $H \rightarrow \infty$, hence so must the number of eigenvalues of our problem which lie in this range.

We can give a physical interpretation to these results if we associate with each value of λ a direction θ in accordance with the rule

$$k = \frac{\omega}{c(z)} \sin \theta(z).$$

This allows us to associate a real ray with each mode for $k^2 > 0$; and these rays have the properties required of them by geometrical optics. For example we have

$$k = \frac{\omega}{c(z_1)} \frac{c(z_1)}{c(z)} \sin \theta(z) = \frac{\omega}{c(z_1)} \sin \theta(z_1),$$

with

$$\sin \theta(z_1) = \frac{c(z_1)}{c(z)} \sin \theta(z).$$

Thus the refraction law of geometrical optics holds with θ interpreted as the angle between the ray and the axis of z .

When $-q_{\max} < \lambda < 0$ there is, for each ray, a critical depth at which it is refracted back towards the surface. At depths greater than this the corresponding mode decays exponentially. These are modes that are formed by the constructive interference of signals describing refraction paths in the fluid. They are thus independent of the nature of the bottom provided it is deep enough.

Those modes with $\lambda \geq 0$ ($a^2 \geq k^2 > 0$) correspond to rays which suffer reflections at the bottom. We can thus interpret these modes as being formed by the constructive interference of bottom reflected signals. When $H = \infty$ we see that there is no interference mechanism, and all signal paths are equally to be preferred. To this extent there is a similarity between the continuous spectrum and an outgoing wave (see also the example of section 5 and equations (6) and (7) of section 6).

The form of equation (4) is not valid in the limit $H = \infty$ if equation (3) has a continuous spectrum. We therefore modify (4) by introducing the function $\sigma(k^2)$ which is constant except at the eigenvalues $k^2 = k_n^2$ when it jumps by $1/\int_0^H \phi_n^2 dz$ (that is, $\sigma(k_n^2+0) - \sigma(k_n^2-0) = 1/\int_0^H \phi_n^2 dz$). We can then write the Green's function as the Stieltjes integral

$$G(r; z, z_P) = -\frac{1}{4}i \int_{k^2=-\infty}^{k^2=\infty} H_0^{(2)}(kr) \phi(z, k) \phi(z_P, k) d\sigma.$$

As in section 2 the convention $\sqrt{-k^2} = -i|k|$ must be observed.

The limit of $\sigma(k^2)$ as $H \rightarrow \infty$ is again a step function for k^2 in the discrete spectrum. However, σ is continuous (even differentiable) for k^2 in the continuous spectrum. For proofs of these results see, for example, Coddington and Levinson ((4), chapter 9), or Titchmarsh ((5), chapters 2-3).

4. Pekeris's construction for the Green's function

An alternative method of obtaining the result (4) is to construct $G(r; z, z_P)$ as the sum of the residues of a contour integral in the complex k -plane. These poles are at the zeros of the period equation as defined by Pekeris in (2). We now derive the period equations in the sense of Pekeris

- (a) for the problem as we have stated it; and
- (b) for an outgoing wave condition at $z = \infty$.

Following Pekeris we begin by attempting to find a solution which has a point source singularity at $(r = 0, z = z_P)$. Thus we seek a solution

which behaves like

$$\int_0^{\infty} J_0(kr) \frac{e^{-i\Omega(z-z_P)}}{-i\Omega} k dk \quad \text{as} \quad \sqrt{\{r^2 + (z-z_P)^2\}} \rightarrow 0,$$

$$\Omega^2 = \omega^2 / \{c(z_P)\}^2 - k^2.$$

We try the solution
$$P = \int_0^{\infty} J_0(kr) \phi(z, z_P) k dk.$$

As ϕ satisfies the second-order differential equation (3), we see that the conditions

$$(1) \quad \phi(z = z_P - 0) = \phi(z = z_P + 0)$$

and

$$(2) \quad \frac{d\phi(z = z_P + 0)}{dz} - \frac{d\phi(z = z_P - 0)}{dz} = 2$$

are sufficient to specify the solution uniquely.

We put
$$\phi_1(z) = A_1 M(z) + B_1 N(z) \quad \text{for } z < z_P,$$

and

$$\phi_2(z) = A_2 M(z) + B_2 N(z) \quad \text{for } z > z_P.$$

The boundary conditions then give

$$\phi_1(z) = A\{N(0)M(z) - M(0)N(z)\}$$

and

$$\phi_2(z)$$

$$= B \left\{ \left[N(H) \cos \alpha + \frac{dN(H)}{dz} \sin \alpha \right] M(z) - \left[M(H) \cos \alpha + \frac{dM(H)}{dz} \sin \alpha \right] N(z) \right\}.$$

The matching conditions then give

$$(1) \quad A\{N(0)M(z_P) - M(0)N(z_P)\} -$$

$$- B \left\{ \left[N(H) \cos \alpha + \frac{dN(H)}{dz} \sin \alpha \right] M(z_P) - \right.$$

$$\left. - \left[M(H) \cos \alpha + \frac{dM(H)}{dz} \sin \alpha \right] N(z_P) \right\} = 0$$

and

$$(2) \quad A \left\{ N(0) \frac{dM(z_P)}{dz} - M(0) \frac{dN(z_P)}{dz} \right\} -$$

$$- B \left\{ \left[N(H) \cos \alpha + \frac{dN(H)}{dz} \sin \alpha \right] \frac{dM(z_P)}{dz} - \right.$$

$$\left. - \left[M(H) \cos \alpha + \frac{dM(H)}{dz} \sin \alpha \right] \frac{dN(z_P)}{dz} \right\} = 2.$$

These equations can be written in matrix form as

$$V\mathbf{x} = \mathbf{b}.$$

The period equation is then

$$\det(V) = 0.$$

Explicitly we have

$$\begin{bmatrix} M(z_P) & N(z_P) \\ \frac{dM(z_P)}{dz} & \frac{dN(z_P)}{dz} \end{bmatrix} \begin{bmatrix} N(0) & N(H)\cos\alpha + \frac{dN(H)}{dz}\sin\alpha \\ -M(0) & -M(H)\cos\alpha - \frac{dM(H)}{dz}\sin\alpha \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

We see that the first matrix is the matrix of the Wronskian of the differential equation (3) and hence is non-singular as M and N are a fundamental set of solutions; and that the second matrix is the matrix of the determinant whose zeros (as functions of k^2) are the eigenvalues.

Thus we can always solve for A and B unless k^2 is an eigenvalue of equation (3). This establishes the equivalence of the zeros of the period equation and the eigenvalues in this case.

In the case of an outgoing wave at $z = \infty$ we have

$$\phi_2(z) = B_2 N(z),$$

and the matching conditions lead to the matrix equation

$$\begin{bmatrix} M(z_P) & N(z_P) \\ \frac{dM(z_P)}{dz} & \frac{dN(z_P)}{dz} \end{bmatrix} \begin{bmatrix} N(0) & 0 \\ -M(0) & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

We see that the characteristic equation is

$$N(0) = 0,$$

which is the limit of the characteristic equation for the boundary conditions (a) as $H \rightarrow \infty$ provided that $-q_{\max} < \lambda < 0$. Thus we again recover the modes associated with the refracted ray paths. The outgoing wave condition is necessarily set on an infinite range so that it is of interest to ask what, in this case, corresponds to the continuous spectrum. Pekeris, who set the outgoing wave condition in his work, found, in addition to the refracted modes, an integral term to which he gave the name of 'branch line integral'. In the next section we reproduce as an example one of the problems solved by Pekeris, and show that, in this case, the branch line integral is identical with the term to which the continuous spectrum gives rise.

Our approach in this section has been somewhat more general than that of Pekeris's original work and is due in part to Haskell, see Ewing (6)—this reference also contains a full summary of Pekeris's results. The use of contour integration to derive eigenfunction expansions is described in detail in Titchmarsh (5).

5. On a problem solved by Pekeris

When the density ρ is allowed to vary with z , equation (3) becomes

$$\frac{d}{dz} \left(\frac{1}{\rho} \frac{d\phi}{dz} \right) + \frac{1}{\rho} \left(\frac{\omega^2}{c^2(z)} - k^2 \right) \phi = 0.$$

This equation is again self-adjoint; and our previous arguments can be applied to it. However, in this case, the orthogonality relation becomes

$$\int_0^H \frac{1}{\rho} \phi_i \phi_j dz = \delta_{ij},$$

and equation (4) becomes

$$G(r; z, z_P) = \frac{-i}{4} \sum_n H_0^{(2)}(k_n r) \frac{\phi_n(z) \phi_n(z_P)}{\rho(z_P)}.$$

Also, if ρ is discontinuous at certain depths, then we must have the additional conditions

$$\phi \text{ continuous, } \frac{1}{\rho} \frac{d\phi}{dz} \text{ continuous}$$

satisfied at these points. These are the Weierstrass-Erdmann corner conditions; see Courant-Hilbert (7). They correspond to continuity of pressure and normal velocity at the layer interface.

We will consider the special model where both ρ and c have step discontinuities at $z = D$ and are otherwise constant. Our boundary conditions are $P = 0$ at $z = 0$ and $z = H$. This problem was solved by Pekeris (2) for the case $H = \infty$, and then used for interpreting the results of explosive sound transmissions in shallow water over a fluid bottom.

The eigenfunctions are given by

$$\phi = A \sin \sqrt{\left(\frac{\omega^2}{c_1^2} - k^2 \right) z} = A \sin \Omega_1 z \quad \text{in layer (1),}$$

$$\text{and } \phi = B \sin \sqrt{\left(\frac{\omega^2}{c_2^2} - k^2 \right) (H - z)} = B \sin \Omega_2 (H - z) \quad \text{in layer (2).}$$

The condition that the matching conditions be satisfied leads to the characteristic equation

$$\begin{vmatrix} \sin \Omega_1 D & \sin \Omega_2 (H - D) \\ \frac{1}{\rho_1} \Omega_1 \cos \Omega_1 D & -\frac{1}{\rho_2} \Omega_2 \cos \Omega_2 (H - D) \end{vmatrix} = 0.$$

For $\omega^2/c_1^2 > k^2 > \omega^2/c_2^2$ the spectrum is discrete in the limit $H = \infty$. The characteristic equation in this case is

$$\tan \Omega_1 D = -\frac{\rho_2 \Omega_1}{\rho_1 \Omega_2} \tanh \Omega_2 (H - D)$$

which tends to $\tan \Omega_1 D = -\frac{\rho_2 \Omega_1}{\rho_1 \Omega_2}$, as $H \rightarrow \infty$.

This equation has discrete solutions if it has solutions at all. The limiting form of the eigenfunction in layer (2) is given by

$$\lim_{H \rightarrow \infty} \frac{\rho_2 \sinh |\Omega_2| (H-z)}{|\Omega_2| \cosh |\Omega_2| (H-D)} \rightarrow \frac{\rho_2 e^{\Omega_2 (D-z)}}{|\Omega_2|}.$$

The spectrum is continuous in the limit $H = \infty$ if $\omega^2/c_2^2 > k^2$. We can write the characteristic equation as

$$\frac{\tan \Omega_2 H - \tan \Omega_2 D}{1 + \tan \Omega_2 H \tan \Omega_2 D} = -\frac{\rho_1 \Omega_2 \tan \Omega_1 D}{\rho_2 \Omega_1}.$$

We see that, if $1 + \tan \Omega_2 H \tan \Omega_2 D = 0$, then it vanishes again for

$$\bar{\Omega}_2 = \Omega_2 + \Delta \Omega_2, \quad \Delta \Omega_2 = \frac{\pi}{H} + O\left(\frac{1}{H^2}\right).$$

Now exactly one eigenvalue lies in the range $(\Omega_2, \bar{\Omega}_2)$. Thus the number of eigenvalues lying in any finite range $(\Omega_2, \Omega_2 + \Delta \Omega_2)$ tends to infinity with $\Delta \Omega_2 H/\pi$, that is with H .

It is of interest to verify that the eigenfunctions are orthogonal on $(0, H)$ with respect to the weight function $1/\rho$ (as they should be). This orthogonality property eluded Pekeris, a fact on which he comments at some length.

Our eigenfunctions have the form

$$\phi = \frac{\rho_1 \sin \Omega_1 z}{\Omega_1 \cos \Omega_1 D} \quad \text{in layer (1),}$$

and
$$\phi = \frac{\rho_2 \sin \Omega_2 (H-z)}{\Omega_2 \cos \Omega_2 (H-D)} \quad \text{in layer (2).}$$

We put
$$I_1 = \int_0^D \frac{1}{\rho_1} \phi_i \phi_j dz, \quad \text{and} \quad I_2 = \int_D^H \frac{1}{\rho_2} \phi_i \phi_j dz.$$

Then

$$I_1 = \frac{1}{\Omega_{1j}^2 - \Omega_{1i}^2} \frac{\rho_1}{\Omega_{1i} \Omega_{1j} \cos \Omega_{1i} D \cos \Omega_{1j} D} \times \\ \times \{\Omega_{1i} \cos \Omega_{1i} D \sin \Omega_{1j} D - \Omega_{1j} \sin \Omega_{1i} D \cos \Omega_{1j} D\}$$

and

$$I_2 = \frac{-1}{\Omega_{2j}^2 - \Omega_{2i}^2} \frac{\rho_2}{\Omega_{2i} \Omega_{2j} \cos \Omega_{2i} (H-D) \cos \Omega_{2j} (H-D)} \times \\ \times \{\Omega_{2i} \cos \Omega_{2i} (H-D) \sin \Omega_{2j} (H-D) - \Omega_{2j} \sin \Omega_{2i} (H-D) \cos \Omega_{2j} (H-D)\}.$$

Thus

$$I_1 = \frac{\phi_j(D) - \phi_i(D)}{\Omega_{1j}^2 - \Omega_{1i}^2} = \frac{\phi_j(D) - \phi_i(D)}{k_i^2 - k_j^2},$$

$$\text{and} \quad I_2 = -\frac{\phi_j(D) - \phi_i(D)}{\Omega_{2j}^2 - \Omega_{2i}^2} = -\frac{\phi_j(D) - \phi_i(D)}{k_i^2 - k_j^2}.$$

$$\text{Hence} \quad I_1 + I_2 = 0 \quad (i \neq j).$$

We see that the role of the weight function is essential to the argument.

In this case the spectral function can be constructed; and thus we can give explicitly the limiting form of equation (4). However, we cannot use the representation

$$\phi = \frac{\rho_2 \sin \Omega_2 (H - z)}{\Omega_2 \cos \Omega_2 (H - D)}$$

for the solution in layer (2) as this expression is indeterminate in the limit $H = \infty$. A suitable form is

$$\phi = \frac{\rho_1 \Omega_2 \sin \Omega_1 D \cos \Omega_2 (z - D) + \rho_2 \Omega_1 \cos \Omega_1 D \sin \Omega_2 (z - D)}{\Omega_2 \Omega_1 \cos \Omega_1 D}.$$

For large values of H we have

$$\int_0^H \frac{1}{\rho} \phi^2 dz = \frac{H}{2} \left[\frac{\rho_1^2 \Omega_2^2 \sin^2 \Omega_1 D + \rho_2^2 \Omega_1^2 \cos^2 \Omega_1 D}{\rho_2 \Omega_1^2 \Omega_2^2 \cos^2 \Omega_1 D} \right] + O(D/H).$$

Also, for large values of H , the distance between adjacent eigenvalues is given by

$$(\Delta \Omega_2)H = \pi + O(1/H).$$

Thus we have

$$\frac{\Delta \sigma(\Omega_2)}{\Delta \Omega_2} = \frac{H}{\pi} \left[\frac{\rho_1^2 \Omega_2^2 \sin^2 \Omega_1 D + \rho_2^2 \Omega_1^2 \cos^2 \Omega_1 D}{\rho_2 \Omega_1^2 \Omega_2^2 \cos^2 \Omega_1 D} \right]^{-1} + O(D/H).$$

$$\text{Hence} \quad \frac{d\sigma(\Omega_2)}{d\Omega_2} = \frac{2}{\pi} \frac{\rho_2 \Omega_1^2 \Omega_2^2 \cos^2 \Omega_1 D}{\rho_1^2 \Omega_2^2 \sin^2 \Omega_1 D + \rho_2^2 \Omega_1^2 \cos^2 \Omega_1 D}.$$

Let Φ denote the limiting form of that part of (4) which belongs to the continuous spectrum. Assuming that z_P belongs to layer I, we have in this layer

$$\Phi = \frac{i}{2\pi} \int_{-\infty}^{\omega/c_2} H_0^{(2)}(kr) \frac{\rho_1 \rho_2 \Omega_2 \sin \Omega_1 z \sin \Omega_1 z_P}{\rho_1^2 \Omega_2^2 \sin^2 \Omega_1 D + \rho_2^2 \Omega_1^2 \cos^2 \Omega_1 D} k dk.$$

For terms in the discrete spectrum we have

$$\begin{aligned} \int_0^\infty \frac{1}{\rho} \phi^2 dz &= \int_0^D \frac{1}{\rho_1} \left(\frac{\rho_1^2 \sin^2 \Omega_1 z}{\Omega_1^2 \cos^2 \Omega_1 D} \right) dz + \int_D^\infty \frac{1}{\rho_2} \left(\frac{\rho_2^2 e^{2\Omega_2(D-z)}}{\Omega_2^2} \right) dz \\ &= \frac{\rho_1}{\Omega_1^2 \cos^2 \Omega_1 D} \left(\frac{D}{2} - \frac{\sin 2\Omega_1 D}{4\Omega_1} \right) + \frac{\rho_2}{2\Omega_2^3} \\ &= \frac{\rho_1}{2\Omega_1^3 \cos^2 \Omega_1 D} \left(\Omega_1 D - \sin \Omega_1 D \cos \Omega_1 D + \frac{\rho_2 \Omega_1^3}{\rho_1 \Omega_2^3} \cos^2 \Omega_1 D \right). \end{aligned}$$

Since

$$\tan \Omega_1 D = -\frac{\rho_2 \Omega_1}{\rho_1 \Omega_2}$$

we get

$$\int_0^\infty \frac{1}{\rho} \phi^2 dz = \frac{\rho_1}{2\Omega_1^3 \cos^2 \Omega_1 D} \times \\ \times \{\Omega_1 D - \sin \Omega_1 D \cos \Omega_1 D - (\rho_1/\rho_2)^2 \sin^2 \Omega_1 D \tan \Omega_1 D\}.$$

Thus the limiting form of (4) is

$$G(r; z, z_P) \\ = -\frac{1}{2}i \sum_n H_0^{(2)}(k_n r) \frac{\Omega_1 \sin \Omega_1 z \sin \Omega_1 z_P}{\Omega_1 D - \sin \Omega_1 D \cos \Omega_1 D - (\rho_1/\rho_2)^2 \sin^2 \Omega_1 D \tan \Omega_1 D} + \\ + \frac{i}{2\pi} \int_{-i\infty}^{\omega/c_2} H_0^{(2)}(kr) \frac{\rho_1 \rho_2 \Omega_2 \sin \Omega_1 z \sin \Omega_1 z_P}{\rho_1^2 \Omega_2^2 \sin^2 \Omega_1 D + \rho_2^2 \Omega_1^2 \cos^2 \Omega_1 D} k dk.$$

The form of the solution given by Pekeris for this problem is

$$4\pi G(r; z, z_P).$$

We can explain the factor 4π if we note that we have effectively adopted the form $(1/4\pi R)e^{-i\omega R/c}$ ($R^2 = r^2 + (z - z_P)^2$) for the singularity at $z = z_P$ while that used by Pekeris is there equivalent to $(1/R)e^{-i\omega R/c}$.

6. Propagation of a transient pulse

For completeness we sketch how our solutions can be modified to describe the propagation of a transient pulse. Let $g(\omega)$ be the Fourier transform of this pulse. Then the required solution is given by

$$P(r, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) G(r; z, z_P, \omega) e^{i\omega t} d\omega,$$

where we consider our Green's function as a function of the parameter ω .

Let us assume that our observations are made at a sufficiently great distance r so that we may use the asymptotic representations of the Hankel function

$$H_0^{(2)}(k_n r) \sim \sqrt{\left(\frac{2}{\pi k_n r}\right)} e^{-i(\frac{1}{4}\pi + k_n r)}.$$

We then have

$$P \sim \frac{-i}{8\pi} \sum_n \int_{-\infty}^{\infty} g(\omega) \sqrt{\left(\frac{2}{\pi k_n r}\right)} \phi_n(z) \phi_n(z_P) e^{-i(\frac{1}{4}\pi + k_n r - \omega t)} d\omega.$$

In practice the major contributions to these integrals are near the points of stationary phase of the exponential terms (see, for instance, Lamb (8))

where the ϕ_n are smoothly varying functions of ω . In this case we can make use of the asymptotic formula

$$\int_{-\infty}^{\infty} \phi(x) e^{if(x)} dx \sim \frac{\sqrt{\pi} \phi(\bar{x})}{\sqrt{(|\frac{1}{2} f''(\bar{x})|)}} e^{i(f(\bar{x}) \pm \frac{1}{2}\pi)},$$

where we take $\pm \frac{1}{2}\pi$ according to the sign of $f''(\bar{x})$, and \bar{x} is the (here assumed unique) point of stationary phase (that is $f'(\bar{x}) = 0$).

In our problem the points of stationary phase are given by

$$\frac{d}{d\omega} \{\omega t - k_n r\} = 0,$$

that is by

$$r = \frac{d\omega}{dk_n} t = {}^nC_g t,$$

where nC_g is the group velocity of the n th mode.

The equation

$$r = {}^nC_g t$$

determines (for a fixed value of r) the arrival times of the different frequency components of the n th mode (here considered as a function of ω , and characterized by the property that ϕ_n vanishes n times in the interior of $(0, H)$ for all values of ω). Thus we see that the arrivals at a given point at a given instant are made up of components from different modes which have the same group velocity. In general each of the arrivals will depend on distinct values of ω (generally different for each mode) unless C_g , as a function of ω , is stationary for this value of ω . When this occurs one would expect some degree of reinforcement. This has been called the 'Airy phase' by Pekeris. The method of stationary phase breaks down for the Airy phase. Pekeris has given a formula which is here applicable.

When $d\{{}^nC_g\}/d\omega \neq 0$, the solution is given by

$$P \sim \frac{-i}{4\pi r} \sum_n g(\omega) \left(\frac{1}{k_n} \left| \frac{d\{{}^nC_g\}}{d\omega} \right| \right)^{\frac{1}{2}} \phi_n(z) \phi_n(z_P) e^{-t\{(\frac{1}{2}\pi \mp \frac{1}{2}\pi + k_n r - \omega t)\}} \quad (5)$$

where the summation is taken over those values of ω such that

$$r = {}^nC_g t, \quad \text{and} \quad k_n^2 > 0.$$

The computation of P from formula (5) requires, in general, the numerical solution of equation (3) for, at any rate, the first few eigenvalues and eigenfunctions for a set of values of ω within the desired frequency range—any other values can then be filled in by interpolation. This is a perfectly practical task using modern computing methods. To compute $d\{{}^nC_g\}/d\omega$ it is first necessary to know nC_g , and this can be determined

numerically by the formula

$$nC_g = k_n \frac{\int_0^H \phi_n^2(z) dz}{\int_0^H [\omega/\{c(z)\}^2] \phi_n^2(z) dz}, \quad (6)$$

provided we know k_n^2 and ϕ_n .

To derive this we differentiate equation (3) with respect to ω . This gives

$$\frac{d^2}{dz^2} \left(\frac{\partial \phi_n}{\partial \omega} \right) + \left\{ \frac{\omega^2}{\{c(z)\}^2} - k_n^2 \right\} \frac{\partial \phi_n}{\partial \omega} = \left\{ 2k_n \frac{dk_n}{d\omega} - \frac{2\omega}{\{c(z)\}^2} \right\} \phi_n.$$

As the boundary conditions hold for all ω we must have the right-hand side orthogonal to ϕ_n over $(0, H)$ for a solution to be possible. Hence

$$\int_0^H \left\{ k \frac{dk_n}{d\omega} - \frac{\omega}{\{c(z)\}^2} \right\} \phi_n^2 dz = 0.$$

From this expression the formula for the group velocity follows immediately. When k is in the continuous spectrum, we have

$$C_g = k / \frac{\omega}{\{c(\infty)\}^2}$$

(as $\int_0^H \phi^2 dz$ is divergent in limit $H = \infty$ for this value of k).

Putting $k = \{\omega/c(z)\} \sin \theta(z)$ we see that

$$C_g|_{z=\infty} = c(\infty) \sin \theta(\infty). \quad (7)$$

That is, the group velocity ultimately corresponds to the horizontal component of the velocity along the ray corresponding to the given value of k . This provides further evidence for the identification of the continuous spectrum with an outgoing wave.

Formula (6) is perhaps most valuable if difference methods are used to solve the eigenvalue problem (3) for k in the discrete spectrum. See, for instance, Fox (9), or Kopal (10). Some recent developments of the methods described in (9) have been given by Osborne (11).

I am indebted to a referee for bringing to my notice some recent work on geometric diffraction theory. He has suggested that these methods would be of value in obtaining numerical results. My impression is that a direct construction of the geometric approximation is useful only before the shape of the pulse becomes distorted by dispersion (that is at short ranges), while (5) is most valuable after dispersion has separated out the various mode terms. Thus, the two approaches would seem to be complementary. In one of the references (Friedlander (12), chapter 3, sections

8-9) some results linking geometrical acoustics with the theory of harmonic wave trains are given. Although this theory serves to unify the two approaches, in practice (see (12), chapter 6) this often means that a programme similar to the one we have adopted is used.

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FINITE DIFFERENCES ASSOCIATED WITH SECOND-ORDER DIFFERENTIAL EQUATIONS

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SUMMARY

Finite-difference approximations to second-order differential equations are generally less satisfactory when terms involving first derivatives are present. We here discuss some methods of approximation which are aimed at improving the accuracy in such cases. They arise out of a closer investigation of a method originally proposed by D. N. de G. Allen and R. V. Southwell for dealing with a partial differential equation which occurs in the field of viscous fluid motion.

The paper falls into three fairly distinct parts. In the first we examine the basic approximation and we are then led to consider whether equally satisfactory results cannot be obtained in certain cases when first-derivative terms are not present. Both ordinary and partial differential equations are considered up to this point; we then make an extension of the basic theory of more especial interest in the case of ordinary differential equations.

1. Introduction

In a recent paper, Allen and Southwell (1) have described a new treatment by relaxation methods of a partial differential equation which occurs in the field of viscous fluid motion. With slight changes in notation the equation is

$$\frac{1}{R} \left(\frac{\partial^2 \zeta}{\partial x^2} - R \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} \right) + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y} = 0, \quad (1)$$

where ζ is the vorticity, ψ the stream function, R the Reynolds number, and x and y orthogonal coordinates. In the numerical treatment of (1) the function ψ may be regarded as a known function of x and y (it is in fact found from a second equation to be solved jointly with (1)). For convenience we write $\partial \psi / \partial x = f(x, y)$ and $\partial \psi / \partial y = g(x, y)$. Then the standard relaxation treatment, using a square mesh of side h with points numbered as in Fig. 1, is to replace the ζ -derivatives at 0 by their finite-difference equivalents (Southwell (2)) which leads, with neglect of terms of order h^4 , to the approximation

$$\left(\frac{1}{R} - \frac{1}{2} g_0 h \right) \zeta_1 + \left(1 + \frac{1}{2} f_0 h \right) \zeta_2 + \left(\frac{1}{R} + \frac{1}{2} g_0 h \right) \zeta_3 + \left(1 - \frac{1}{2} f_0 h \right) \zeta_4 - 2 \left(1 + \frac{1}{R} \right) \zeta_0 = 0. \quad (2)$$

The aim of the relaxation process is to satisfy (2) at each nodal point 0 to some degree of accuracy acceptable as error (residual error). This can be made as small as we please for a given mesh size but the solution possesses

also the more inherent error due to the finite-difference approximation. If $f(x, y)$ and $g(x, y)$ are numerically large or vary rapidly this error can be serious except on a very fine mesh. Allen and Southwell suggest that this is so in their case and have devised the following alternative approximation to (1), aimed at overcoming the difficulty. In the neighbourhood of 0 the

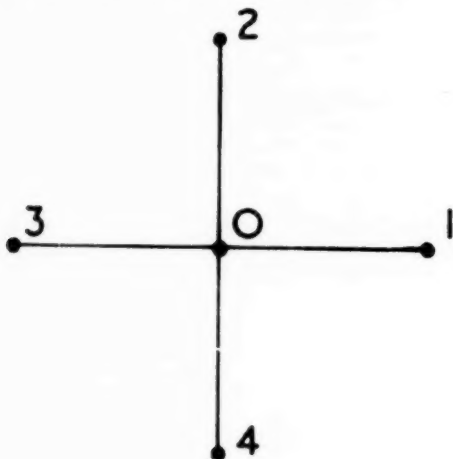


FIG. 1

functions $f(x, y)$ and $g(x, y)$ are replaced by their nodal values f_0 and g_0 and (1) is separated into the two equations

$$\left. \begin{aligned} \frac{\partial^2 \zeta}{\partial x^2} - Rg_0 \frac{\partial \zeta}{\partial x} &= -RA \\ \frac{\partial^2 \zeta}{\partial y^2} + f_0 \frac{\partial \zeta}{\partial y} &= A \end{aligned} \right\}, \quad (3)$$

where A is constant. The solution of the first of (3) is

$$g_0 \zeta = Ax + P + Qe^{Rg_0 x}, \quad (4)$$

where P and Q are integration constants. Using (4) as an expansion along the x -line 301 we can dispose of P and Q and express A in terms of nodal values of ζ , thus:

$$g_0\{(\zeta_1 - \zeta_0)e^{-Rg_0 h} + \zeta_3 - \zeta_0\} = Ah(e^{-Rg_0 h} - 1). \quad (5)$$

Dealing similarly with the second equation of (3), which likewise yields an expansion to be used along the y -line 402, we obtain

$$f_0\{(\zeta_2 - \zeta_0)e^{f_0 h} + \zeta_4 - \zeta_0\} = Ah(e^{f_0 h} - 1). \quad (6)$$

Equating the values of A in (5) and (6) gives the equation

$$g_0 h \{ (\zeta_1 - \zeta_0) e^{-Rg_0 h} + \zeta_3 - \zeta_0 \} / (1 - e^{-Rg_0 h}) - f_0 h \{ (\zeta_2 - \zeta_0) e^{fg_0 h} + \zeta_4 - \zeta_0 \} / (1 - e^{fg_0 h}) = 0, \quad (7)$$

which Allen and Southwell find to be a superior nodal approximation to (2). It is clear that any superiority of (7) over (2) must lie in the approximation it makes to the first-derivative terms in (1), for it is easily shown that the two formulae are identical when these terms are absent. It is primarily this question that we investigate in this paper, i.e. we attempt to discover the reason for the superiority and to point out any advantages in this type of approach which is undoubtedly new in finite-difference theory.

2. It is convenient to start with the ordinary differential equation

$$\zeta'' + p(x)\zeta' + q(x)\zeta = r(x), \quad (8)$$

where primes denote differentiation with respect to x , and subsequently to make the extension to partial equations. There are, in any case, some points of special interest for ordinary equations. The foregoing process would replace (8) along the line 301 by the equation

$$\zeta'' + p_0 \zeta' + q_0 \zeta = r_0, \quad (9)$$

so that
$$\zeta = \frac{r_0}{q_0} + e^{-\frac{1}{2}p_0 x} (P \cosh k_0 x + Q \sinh k_0 x), \quad (10)$$

where
$$k_0 = (\frac{1}{4}p_0^2 - q_0)^{\frac{1}{2}}. \quad (11)$$

Eliminating P and Q in terms of nodal values of ζ gives

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - 2\zeta_0 \cosh k_0 h - \frac{2r_0}{q_0} (\cosh \frac{1}{2}p_0 h - \cosh k_0 h) = 0, \quad (12)$$

which is sufficient to define a set of nodal approximations to the one-dimensional solution of (8). Equation (12) is valid whether k_0 is real or imaginary. It holds also when $q_0 = 0$ (although (10) must now be modified) and if we take the appropriate limit of the last term we obtain

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - 2\zeta_0 \cosh \frac{1}{2}p_0 h - \frac{2r_0 h}{p_0} \sinh \frac{1}{2}p_0 h = 0, \quad (13)$$

which is the same as (5) when applied to the first equation of (3) (where A would be given if this were an independent ordinary equation). Now the standard approximation to (8) derives from the central-difference formulae, in usual notation (Fox (3))

$$h\zeta_0' = (\mu\delta - \frac{1}{6}\mu\delta^3 + \frac{1}{30}\mu\delta^5 - \frac{1}{420}\mu\delta^7 + \dots)\zeta_0 \quad (14)$$

and
$$h^2\zeta_0'' = (\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \frac{1}{560}\delta^8 + \dots)\zeta_0, \quad (15)$$

and the customary three-point formula

$$(1 + \frac{1}{2}p_0 h)\zeta_1 + (1 - \frac{1}{2}p_0 h)\zeta_3 - (2 - h^2q_0)\zeta_0 - h^2r_0 = 0 \quad (16)$$

is obtained by neglecting differences beyond the second in (14) and (15). The reason that it is sometimes unsatisfactory when the first-derivative term is important is evident from the much slower rate of convergence of the series (14) as compared with (15) (which is usually highly satisfactory). A possible way of getting over the difficulty is to remove the first-derivative term in (8) by the usual substitution

$$\zeta = Ye^{-\frac{1}{2}\int p \, dx}, \quad (17)$$

where the new dependent variable satisfies the equation

$$Y'' + \phi Y = re^{\frac{1}{2}\int p \, dx}, \quad (18)$$

and

$$\phi(x) = q - \frac{1}{2}p' - \frac{1}{4}p^2. \quad (19)$$

There are sometimes objections to this approach. For example, the new variable may not be well behaved over the range of integration; or if $p(x)$ undergoes large variations, the right-hand side of (18) may vary too rapidly for satisfactory numerical treatment. But both these objections can be removed if we apply (17) *only locally* over the interval 301 and moreover *only temporarily*, i.e. having applied it we return at once to the variable ζ . Moreover, when we do this we find that the resulting approximation bears a very close relationship to Allen and Southwell's, particularly when it is applied to (1) which is, rather fortuitously, a very special case.

3. The basic approximation

We suppose the points 3, 0, 1 (Fig. 1) to be respectively $x_0 - h$, x_0 , and $x_0 + h$ and let (17) be defined in the interval $(x_0 - h, x_0 + h)$. We may choose any value of the indefinite integral so for convenience we take the value which vanishes at $x = x_0$, i.e. let

$$u(x) = \frac{1}{2} \int_{x_0}^x p \, dx \quad (20)$$

so that the corresponding form of (18) is

$$Y'' + \phi Y = re^u. \quad (21)$$

Neglecting terms in h^4 the standard three-point approximation to (21) is

$$Y_1 + Y_3 - (2 - h^2\phi_0)Y_0 - h^2r_0 = 0 \quad (22)$$

and eliminating pivotal values of Y using (17) we have finally

$$\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - (2 - h^2\phi_0)\zeta_0 - h^2r_0 = 0. \quad (23)$$

If p is given as an analytical function of x , and u can be found exactly, then (23) is the final basic approximation. Otherwise we must expand p about $x = x_0$ in the usual way, i.e.

$$p(x) = p_0 + (x - x_0)p'_0 + \frac{1}{2!}(x - x_0)^2 p''_0 + \dots, \quad (24)$$

and a first approximation to u would be

$$u(x) = \frac{1}{2}p_0(x-x_0), \quad (25)$$

giving the corresponding form of (23)

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - (2 - h^2 \phi_0) \zeta_0 - h^2 r_0 = 0. \quad (26)$$

We can express $h^2 \phi_0$ in (26) in the usual finite-difference form, i.e.

$$h^2 \phi_0 = h^2(q_0 - \frac{1}{4}p_0^2) - \frac{h}{4}(p_1 - p_3). \quad (27)$$

At this stage we can usefully compare (26) with (12); the exponential terms in each compare exactly and if we expand the hyperbolic functions in (12) in powers of h as far as terms in h^4 we get, approximately,

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - (2 - h^2 q_0 + \frac{1}{4}h^2 p_0^2) \zeta_0 - h^2 r_0 = 0. \quad (28)$$

Equations (26) and (28) agree only if $p(x)$ is invariant with respect to x , i.e. the last bracket in (27) is identically zero. But there can be little doubt that if p varies even only moderately with x this term will be important; its magnitude could easily exceed that of the other terms in $h^2 \phi_0$ (e.g. in the numerical example which follows). However, in obtaining (26) from (23) we have neglected a similar term of magnitude $\frac{1}{2}h^2 p'_0$ in approximating to u_1 and u_3 and if, as we suggest, the last term in (27) should not be ignored, then we should also retain this term, i.e. we include the term in p'_0 in (24) to give (according to the trapezoidal formula)

$$u_1 = \frac{h}{4}(p_0 + p_1), \quad u_3 = -\frac{h}{4}(p_0 + p_3). \quad (29)$$

More accurate formulae can be obtained by taking into account more terms of (24) (e.g. the three-point formulae obtained by assuming parabolic variation of p over the points 3, 0, and 1).

Example 1

A very simple example will suffice to illustrate the point. The function

$$\zeta = 1 - \operatorname{erf} x, \quad (30)$$

where $\operatorname{erf} x$ is the error function $(2/\sqrt{\pi}) \int_0^x e^{-x^2} dx$, satisfies the equation

$$\zeta'' + 2x\zeta' = 0, \quad (31)$$

with boundary conditions $\zeta(0) = 1$, $\zeta(\infty) = 0$. It is recorded in Table I at intervals $h = 0.4$ and correct to four decimals; also recorded are approximate ζ -values satisfying respectively the standard approximation (16) (with $p_0 = 2x_0$, $q_0 = r_0 = 0$), equation (13), and equation (23) (with u -values calculated from (29), which gives exact values of the integrals in

this case). The approximate solutions were obtained by the usual relaxation methods, i.e. by liquidating residuals, defined by the left-hand side of the appropriate equation, using relaxation patterns. The results adequately confirm the points we have made so far, which we may summarize thus. Although the order of the error term in (16) and (23) appears to be the same, the latter is more satisfactory because it eliminates the effect of the slowly convergent difference series associated with the first-derivative term. Equation (12) (or (13) if appropriate, as in this example) has the same

TABLE 1

	<i>Eqn. (16)</i>	<i>Eqn. (13)</i>	<i>Eqn. (23)</i>	<i>Eqn. (30)</i>
<i>x</i>	ζ	ζ	ζ	ζ
0.0	1.0000	1.0000	1.0000	1.0000
0.4	0.5576	0.5663	0.5726	0.5716
0.8	0.2372	0.2514	0.2583	0.2579
1.2	0.0722	0.0854	0.0898	0.0897
1.6	0.0142	0.0218	0.0238	0.0237
2.0	0.0014	0.0032	0.0047	0.0047
2.4	0.0000	0.0006	0.0007	0.0007
2.8	0.0000	0.0001	0.0001	0.0001

effect but omits an important term arising from the variation of $p(x)$. We can show that (12) is certainly correct to the same order of accuracy as (16). Expanding the exponential and hyperbolic functions in (12) we obtain

$$(1 + \frac{1}{2}p_0 h)\zeta_1 + (1 - \frac{1}{2}p_0 h)\zeta_3 - (2 - h^2 q_0)\zeta_0 - h^2 r_0 + \frac{1}{8}p_0^2 h^2 (\zeta_1 + \zeta_3 - 2\zeta_0) + \frac{1}{48}p_0^3 h^3 (\zeta_1 - \zeta_3) = O(h^4) \quad (32)$$

and the last two terms on the left-hand side of (32) involve $\delta^2 \zeta_0$ and $\mu \delta \zeta_0$ respectively and so, by (14) and (15), are each of order h^4 ; thus (12) and (16) differ by terms of order h^4 (but whose magnitude may be large if p_0 is large).

4. To deal with the partial differential equation (1) we proceed in a similar manner to that used by Allen and Southwell, separating it into the two equations

$$\left. \begin{aligned} \frac{\partial^2 \zeta}{\partial x^2} - Rg(x, y) \frac{\partial \zeta}{\partial x} &= -RA(x, y) \\ \text{and} \quad \frac{\partial^2 \zeta}{\partial y^2} + f(x, y) \frac{\partial \zeta}{\partial y} &= A(x, y) \end{aligned} \right\} \quad (33)$$

Along the x -line 301 we have $g(x, y) = g(x, y_0)$ and $A(x, y) = A(x, y_0)$, where 0 is (x_0, y_0) , and the solution for ζ is a function of x alone. Along the y -line 402, $f(x, y) = f(x_0, y)$, $A(x, y) = A(x_0, y)$ and ζ is a function of y alone.

Applying the process of the last section to each of (33) in turn and eliminating $A_0 (= A(x_0, y_0))$ from the two resulting equations we obtain as an approximation to (1) at 0 the equation

$$\frac{1}{R}(\zeta_1 e^{u_1} + \zeta_3 e^{u_3}) + \zeta_2 e^{v_2} + \zeta_4 e^{v_4} - \left(2 + \frac{2}{R} - h^2 \Phi_0\right) \zeta_0 = 0, \quad (34)$$

where

$$\left. \begin{aligned} u(x) &= -\frac{1}{2} R \int_{x_0}^x g(x, y_0) dx \\ v(y) &= \frac{1}{2} \int_{y_0}^y f(x_0, y) dy \end{aligned} \right\}, \quad (35)$$

$$\text{and} \quad \Phi_0 = \frac{1}{2} \left(\frac{\partial g}{\partial x} \right)_0 - \frac{1}{2} \left(\frac{\partial f}{\partial y} \right)_0 - \frac{1}{4} (f_0^2 + R g_0^2). \quad (36)$$

To obtain some comparison between (34) and (7) we can proceed as in the previous section. Equation (7) can be written in the equivalent form obtained by combining two equations of type (13) (one in the x -direction and one in the y -direction) and we can expand each of these in the form (28) (with appropriate changes in symbols). Thus combining these last two we obtain

$$\frac{1}{R}(\zeta_1 e^{-\frac{1}{2} R g_0 h} + \zeta_3 e^{\frac{1}{2} R g_0 h}) + \zeta_2 e^{\frac{1}{2} f_0 h} + \zeta_4 e^{-\frac{1}{2} f_0 h} - \left\{ 2 + \frac{2}{R} + \frac{h^2}{4} (f_0^2 + R g_0^2) \right\} \zeta_0 = 0 \quad (37)$$

as a close approximation to (7), i.e. with neglect of terms in h^4 . To find expressions for pivotal values of u and v we proceed as in the last section by expanding $g(x, y_0)$ along the x -line 301 and likewise expanding $f(x_0, y)$ along the y -line 402. Substituting the corresponding first approximations found in this way into (34) this equation becomes

$$\frac{1}{R}(\zeta_1 e^{-\frac{1}{2} R g_0 h} + \zeta_3 e^{\frac{1}{2} R g_0 h}) + \zeta_2 e^{\frac{1}{2} f_0 h} + \zeta_4 e^{-\frac{1}{2} f_0 h} - \left(2 + \frac{2}{R} - h^2 \Phi_0 \right) \zeta_0 = 0, \quad (38)$$

which we can now compare with (37). We find that the equations agree exactly by virtue of the fact that $f(x, y) = \partial \psi / \partial x$, $g(x, y) = \partial \psi / \partial y$, so that the first two terms on the right-hand side of (36) cancel each other identically. From the point of view of the numerical analysis this is rather fortuitous; in the general case f and g may make an important contribution to (36) which our analysis suggests should not be ignored. The other general point which emerges is that even in Allen and Southwell's case a substantial improvement in accuracy should result from using the trapezoidal (or an even more accurate) formula to estimate pivotal values of u and v ; this would involve an almost negligible increase in labour.

5. This point leads us to consider whether Allen and Southwell's process could itself be modified to allow of this better approximation. As it stands the method replaces (8) by the equation (9); but if we *first* apply the transformation (17) to (8) and *then* approximate to the resulting equation (21) by the equation

$$Y'' + \phi_0 Y = r_0 \quad (39)$$

we would obtain

$$Y_1 + Y_3 - 2Y_0 \cos \phi_0^{\frac{1}{2}} h - \frac{2r_0}{\phi_0} (1 - \cos \phi_0^{\frac{1}{2}} h) = 0 \quad (40)$$

as an approximation to (21) at each nodal point and hence

$$\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - 2\zeta_0 \cos \phi_0^{\frac{1}{2}} h - \frac{2r_0}{\phi_0} (1 - \cos \phi_0^{\frac{1}{2}} h) = 0 \quad (41)$$

as a nodal approximation to (8), u having the significance of (20). This formula is quite different from either (12) or (23) although it agrees with the latter as far as terms in h^4 . It has two clear advantages over (12) in that we can extend u_1 and u_3 beyond the first approximations $u_1 = \frac{1}{2} p_0 h$, $u_3 = -\frac{1}{2} p_0 h$ (to which (12) is of necessity restricted) and, moreover, it takes proper account of the term involving p' (which occurs in ϕ). Applying this process to each of the equations (33) in turn and combining the results in the manner previously used, we obtain as a nodal approximation to (1) the equation

$$x_0(\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - 2\zeta_0 \cosh \alpha_0^{\frac{1}{2}} h) - R(1 - \cosh \alpha_0^{\frac{1}{2}} h) + \beta_0(\zeta_2 e^{v_2} + \zeta_4 e^{v_4} - 2\zeta_0 \cosh \beta_0^{\frac{1}{2}} h) (1 - \cosh \beta_0^{\frac{1}{2}} h) = 0, \quad (42)$$

where

$$\alpha_0 = -\frac{1}{2} R \left(\frac{c'g}{cx} \right)_0 + \frac{1}{4} R^2 g_0^2 \quad (43)$$

and

$$\beta_0 = \frac{1}{2} \left(\frac{c'f}{cy} \right)_0 + \frac{1}{4} R^2 f_0^2$$

Here u and v have the significance of (35).

It would appear that (42) is a more correct application to (1) of Allen and Southwell's type of process than (7) and the amount of initial computation in using it would be substantially the same. But in deciding its merit relative to (34) an essentially new point is involved. It is whether we may in general expect the approximation (40) to represent (21) more accurately than customary finite differences. There is some evidence that this may be so for if ϕ and c' are invariant with respect to x , (40) would represent (21)

also the more inherent error due to the finite-difference approximation. If

6. Some applications to a class of eigenvalue problems

We shall consider the one-dimensional problem governed by the equation

$$\zeta'' + \{\lambda\rho(x) - \sigma(x)\}\zeta = 0 \quad (44)$$

with extensions to its two-dimensional counterpart

$$\nabla^2\zeta + \{\lambda\rho(x, y) - \sigma(x, y)\}\zeta = 0 \quad \left(\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad (45)$$

where in each case $\rho > 0$ throughout the domain of the problem and λ is a parameter possessing a discrete set of eigenvalues, with a corresponding set of modes for ζ , when ζ is required to satisfy specified boundary conditions (e.g. ζ vanishes on a simple closed boundary in the case of (45) or at the end points in the case of (44)). Both equations have physical importance in many fields.

For (44) the usual difference approximation at a nodal point 0 is

$$\zeta_1 + \zeta_3 - (2 - h^2 q_0)\zeta_0 = 0 \quad (q = \lambda\rho - \sigma) \quad (46)$$

and to find an approximation to a given mode we have to solve (46) for a set of nodal ζ -values together with the corresponding eigenvalue. If an initial approximation to the mode of (46) is known, relaxation methods can usually be employed satisfactorily for this purpose, the eigenvalue λ being estimated at any stage by the Rayleigh quotient which corresponds to (46), i.e.

$$\lambda h^2 = - \frac{\sum \{(\zeta_1 + \zeta_3 - 2\zeta_0)\zeta_0 - h^2\sigma_0\zeta_0^2\}}{\sum \rho_0\zeta_0^2}, \quad (47)$$

where the summations extend over all ζ -values, including the end-points. This method is particularly satisfactory when $\zeta = 0$ at the end points and the mode considered has no internal zeros (corresponding to the smallest λ). The solutions of (46) depend upon h and converge to the true solutions of (44) as h is decreased indefinitely. In a recent paper, Bolton and Scoins (4) have investigated the convergence in the special case $\rho(x) = 1$; (44) then becomes the one-dimensional Schrödinger equation with σ as the potential function. They show that if $\lambda = \Lambda(h)$ denotes the approximation obtained from (46) to a given true λ of (44), then

$$\Lambda(h) = \lambda + \epsilon h^2 + O(h^4) \quad (48)$$

and that if $\zeta = 0$ at the end points and σ is bounded in $(0, 1)$ and non-singular at the end points, then $\epsilon < 0$, i.e. for small enough h , $\Lambda(h) < \lambda$. (5) $\Lambda(h) - \lambda$ is of order h^2 or h^4 according as σ is continuous or not at the end points.

also the more inherent error due to the finite-difference approximation. If $f(x, y)$ and $g(x, y)$ are numerically large or vary rapidly this error can be serious except on a very fine mesh. Allen and Southwell suggest that this is so in their case and have devised the following alternative approximation to (1), aimed at overcoming the difficulty. In the neighbourhood of 0 the

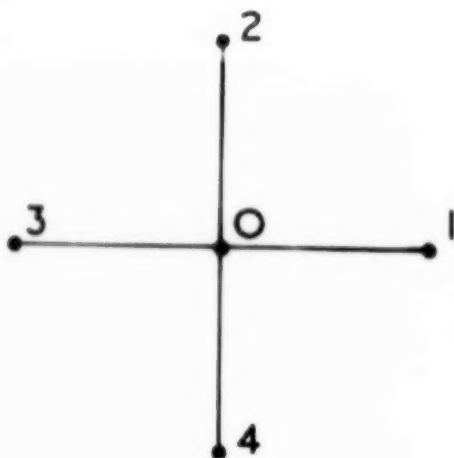


FIG. 1

functions $f(x, y)$ and $g(x, y)$ are replaced by their nodal values f_0 and g_0 and (1) is separated into the two equations

$$\left. \begin{aligned} \frac{\partial^2 \zeta}{\partial x^2} - R g_0 \frac{\partial \zeta}{\partial x} &= -R A \\ \frac{\partial^2 \zeta}{\partial y^2} + f_0 \frac{\partial \zeta}{\partial y} &= A \end{aligned} \right\} \quad (3)$$

where A is constant. The solution of the first of (3) is

$$g_0 \zeta = A x + P + Q e^{R g_0 x}, \quad (4)$$

where P and Q are integration constants. Using (4) as an expansion along the x -line 301 we can dispose of P and Q and express A in terms of nodal values of ζ , thus:

$$g_0 \{ (\zeta_1 - \zeta_0) e^{-R g_0 h} + \zeta_3 - \zeta_0 \} = A h (e^{-R g_0 h} - 1). \quad (5)$$

Dealing similarly with the second equation of (3), which likewise yields an expansion to be used along the y -line 402, we obtain

$$f_0 \{ (\zeta_2 - \zeta_0) e^{f_0 h} + \zeta_4 - \zeta_0 \} = A h (e^{f_0 h} - 1). \quad (6)$$

Equating the values of A in (5) and (6) gives the equation

$$g_0 h \{ (\zeta_1 - \zeta_0) e^{-Rg_0 h} + \zeta_3 - \zeta_0 \} / (1 - e^{-Rg_0 h}) - f_0 h \{ (\zeta_2 - \zeta_0) e^{f_0 h} + \zeta_4 - \zeta_0 \} / (1 - e^{f_0 h}) = 0, \quad (7)$$

which Allen and Southwell find to be a superior nodal approximation to (2). It is clear that any superiority of (7) over (2) must lie in the approximation it makes to the first-derivative terms in (1), for it is easily shown that the two formulae are identical when these terms are absent. It is primarily this question that we investigate in this paper, i.e. we attempt to discover the reason for the superiority and to point out any advantages in this type of approach which is undoubtedly new in finite-difference theory.

2. It is convenient to start with the ordinary differential equation

$$\zeta'' + p(x)\zeta' + q(x)\zeta = r(x), \quad (8)$$

where primes denote differentiation with respect to x , and subsequently to make the extension to partial equations. There are, in any case, some points of special interest for ordinary equations. The foregoing process would replace (8) along the line 301 by the equation

$$\zeta'' + p_0 \zeta' + q_0 \zeta = r_0, \quad (9)$$

so that
$$\zeta = \frac{r_0}{q_0} + e^{-\frac{1}{2}p_0 x} (P \cosh k_0 x + Q \sinh k_0 x), \quad (10)$$

where
$$k_0 = (\frac{1}{4}p_0^2 - q_0)^{\frac{1}{2}}. \quad (11)$$

Eliminating P and Q in terms of nodal values of ζ gives

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - 2\zeta_0 \cosh k_0 h - \frac{2r_0}{q_0} (\cosh \frac{1}{2}p_0 h - \cosh k_0 h) = 0, \quad (12)$$

which is sufficient to define a set of nodal approximations to the one-dimensional solution of (8). Equation (12) is valid whether k_0 is real or imaginary. It holds also when $q_0 = 0$ (although (10) must now be modified) and if we take the appropriate limit of the last term we obtain

$$\zeta_1 e^{\frac{1}{2}p_0 h} + \zeta_3 e^{-\frac{1}{2}p_0 h} - 2\zeta_0 \cosh \frac{1}{2}p_0 h - \frac{2r_0 h}{p_0} \sinh \frac{1}{2}p_0 h = 0, \quad (13)$$

which is the same as (5) when applied to the first equation of (3) (where A would be given if this were an independent ordinary equation). Now the standard approximation to (8) derives from the central-difference formulae, in usual notation (Fox (3))

$$h\zeta_0' = (\mu\delta - \frac{1}{6}\mu\delta^3 + \frac{1}{30}\mu\delta^5 - \frac{1}{420}\mu\delta^7 + \dots)\zeta_0 \quad (14)$$

and
$$h^2\zeta_0'' = (\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \frac{1}{560}\delta^8 + \dots)\zeta_0, \quad (15)$$

and the customary three-point formula

$$(1 + \frac{1}{2}p_0 h)\zeta_1 + (1 - \frac{1}{2}p_0 h)\zeta_3 - (2 - h^2q_0)\zeta_0 - h^2r_0 = 0 \quad (16)$$

is obtained by neglecting differences beyond the second in (14) and (15). The reason that it is sometimes unsatisfactory when the first-derivative term is important is evident from the much slower rate of convergence of the series (14) as compared with (15) (which is usually highly satisfactory). A possible way of getting over the difficulty is to remove the first-derivative term in (8) by the usual substitution

$$\zeta = Y e^{-i \int p \, dx}, \quad (17)$$

where the new dependent variable satisfies the equation

$$Y'' + \phi Y = r e^{i \int p \, dx}, \quad (18)$$

and

$$\phi(x) = q - \frac{1}{2} p' - \frac{1}{4} p^2. \quad (19)$$

There are sometimes objections to this approach. For example, the new variable may not be well behaved over the range of integration; or if $p(x)$ undergoes large variations, the right-hand side of (18) may vary too rapidly for satisfactory numerical treatment. But both these objections can be removed if we apply (17) *only locally* over the interval 301 and moreover *only temporarily*, i.e. having applied it we return at once to the variable ζ . Moreover, when we do this we find that the resulting approximation bears a very close relationship to Allen and Southwell's, particularly when it is applied to (1) which is, rather fortuitously, a very special case.

3. The basic approximation

We suppose the points 3, 0, 1 (Fig. 1) to be respectively $x_0 - h$, x_0 , and $x_0 + h$ and let (17) be defined in the interval $(x_0 - h, x_0 + h)$. We may choose any value of the indefinite integral so for convenience we take the value which vanishes at $x = x_0$, i.e. let

$$u(x) = \frac{1}{2} \int_{x_0}^x p \, dx \quad (20)$$

so that the corresponding form of (18) is

$$Y'' + \phi Y = r e^u. \quad (21)$$

Neglecting terms in h^4 the standard three-point approximation to (21) is

$$Y_1 + Y_3 - (2 - h^2 \phi_0) Y_0 - h^2 r_0 = 0 \quad (22)$$

and eliminating pivotal values of Y using (17) we have finally

$$\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - (2 - h^2 \phi_0) \zeta_0 - h^2 r_0 = 0. \quad (23)$$

If p is given as an analytical function of x , and u can be found exactly, then (23) is the final basic approximation. Otherwise we must expand p about $x = x_0$ in the usual way, i.e.

$$p(x) = p_0 + (x - x_0) p'_0 + \frac{1}{2!} (x - x_0)^2 p''_0 + \dots, \quad (24)$$

and a first approximation to u would be

$$u(x) = \frac{1}{2}p_0(x-x_0), \quad (25)$$

giving the corresponding form of (23)

$$\zeta_1 e^{1/2 p_0 h} + \zeta_3 e^{-1/2 p_0 h} - (2 - h^2 \phi_0) \zeta_0 - h^2 r_0 = 0. \quad (26)$$

We can express $h^2 \phi_0$ in (26) in the usual finite-difference form, i.e.

$$h^2 \phi_0 = h^2 (q_0 - \frac{1}{4} p_0^2) - \frac{h}{4} (p_1 - p_3). \quad (27)$$

At this stage we can usefully compare (26) with (12); the exponential terms in each compare exactly and if we expand the hyperbolic functions in (12) in powers of h as far as terms in h^4 we get, approximately,

$$\zeta_1 e^{1/2 p_0 h} + \zeta_3 e^{-1/2 p_0 h} - (2 - h^2 q_0 + \frac{1}{4} h^2 p_0^2) \zeta_0 - h^2 r_0 = 0. \quad (28)$$

Equations (26) and (28) agree only if $p(x)$ is invariant with respect to x , i.e. the last bracket in (27) is identically zero. But there can be little doubt that if p varies even only moderately with x this term will be important; its magnitude could easily exceed that of the other terms in $h^2 \phi_0$ (e.g. in the numerical example which follows). However, in obtaining (26) from (23) we have neglected a similar term of magnitude $\frac{1}{2} h^2 p'_0$ in approximating to u_1 and u_3 and if, as we suggest, the last term in (27) should not be ignored, then we should also retain this term, i.e. we include the term in p'_0 in (24) to give (according to the trapezoidal formula)

$$u_1 = \frac{h}{4} (p_0 + p_1), \quad u_3 = -\frac{h}{4} (p_0 + p_3). \quad (29)$$

More accurate formulae can be obtained by taking into account more terms of (24) (e.g. the three-point formulae obtained by assuming parabolic variation of p over the points 3, 0, and 1).

Example 1

A very simple example will suffice to illustrate the point. The function

$$\zeta = 1 - \operatorname{erf} x, \quad (30)$$

where $\operatorname{erf} x$ is the error function $(2/\sqrt{\pi}) \int_0^x e^{-x^2} dx$, satisfies the equation

$$\zeta'' + 2x\zeta' = 0, \quad (31)$$

with boundary conditions $\zeta(0) = 1$, $\zeta(\infty) = 0$. It is recorded in Table 1 at intervals $h = 0.4$ and correct to four decimals; also recorded are approximate ζ -values satisfying respectively the standard approximation (16) (with $p_0 = 2x_0$, $q_0 = r_0 = 0$), equation (13), and equation (23) (with u -values calculated from (29), which gives exact values of the integrals in

this case). The approximate solutions were obtained by the usual relaxation methods, i.e. by liquidating residuals, defined by the left-hand side of the appropriate equation, using relaxation patterns. The results adequately confirm the points we have made so far, which we may summarize thus. Although the order of the error term in (16) and (23) appears to be the same, the latter is more satisfactory because it eliminates the effect of the slowly convergent difference series associated with the first-derivative term. Equation (12) (or (13) if appropriate, as in this example) has the same

TABLE I

x	<i>Eqn. (16)</i>	<i>Eqn. (13)</i>	<i>Eqn. (23)</i>	<i>Eqn. (30)</i>
	ζ	ζ	ζ	ζ
0.0	1.0000	1.0000	1.0000	1.0000
0.4	0.5576	0.5663	0.5726	0.5716
0.8	0.2372	0.2514	0.2583	0.2579
1.2	0.0722	0.0854	0.0898	0.0897
1.6	0.0142	0.0218	0.0238	0.0237
2.0	0.0014	0.0042	0.0047	0.0047
2.4	0.0000	0.0006	0.0007	0.0007
2.8	0.0000	0.0001	0.0001	0.0001

effect but omits an important term arising from the variation of $p(x)$. We can show that (12) is certainly correct to the same *order* of accuracy as (16). Expanding the exponential and hyperbolic functions in (12) we obtain

$$(1 + \frac{1}{2}p_0h)\zeta_1 + (1 - \frac{1}{2}p_0h)\zeta_3 - (2 - h^2q_0)\zeta_0 - h^2r_0 + \frac{1}{8}p_0^2h^2(\zeta_1 + \zeta_3 - 2\zeta_0) + \frac{1}{48}p_0^3h^3(\zeta_1 - \zeta_3) = O(h^4) \quad (32)$$

and the last two terms on the left-hand side of (32) involve $\delta^2\zeta_0$ and $\mu\delta\zeta_0$ respectively and so, by (14) and (15), are each of order h^4 ; thus (12) and (16) differ by terms of order h^4 (but whose magnitude may be large if p_0 is large).

4. To deal with the partial differential equation (1) we proceed in a similar manner to that used by Allen and Southwell, separating it into the two equations

$$\left. \begin{aligned} \frac{\partial^2 \zeta}{\partial x^2} - Rg(x, y) \frac{\partial \zeta}{\partial x} &= -RA(x, y) \\ \text{and} \quad \frac{\partial^2 \zeta}{\partial y^2} + f(x, y) \frac{\partial \zeta}{\partial y} &= A(x, y) \end{aligned} \right\} \quad (33)$$

Along the x -line 301 we have $g(x, y) = g(x, y_0)$ and $A(x, y) = A(x, y_0)$, where 0 is (x_0, y_0) , and the solution for ζ is a function of x alone. Along the y -line 402, $f(x, y) = f(x_0, y)$, $A(x, y) = A(x_0, y)$ and ζ is a function of y alone.

Applying the process of the last section to each of (33) in turn and eliminating $A_0 (= A(x_0, y_0))$ from the two resulting equations we obtain as an approximation to (1) at 0 the equation

$$\frac{1}{R}(\zeta_1 e^{u_1} + \zeta_3 e^{u_3}) + \zeta_2 e^{v_2} + \zeta_4 e^{v_4} - \left(2 + \frac{2}{R} - h^2 \Phi_0\right) \zeta_0 = 0, \quad (34)$$

where

$$\left. \begin{aligned} u(x) &= -\frac{1}{2} R \int_{x_0}^x g(x, y_0) dx \\ v(y) &= \frac{1}{2} \int_{y_0}^y f(x_0, y) dy \end{aligned} \right\}, \quad (35)$$

and

$$\Phi_0 = \frac{1}{2} \left(\frac{\partial g}{\partial x} \right)_0 - \frac{1}{2} \left(\frac{\partial f}{\partial y} \right)_0 - \frac{1}{4} (f_0^2 + Rg_0^2). \quad (36)$$

To obtain some comparison between (34) and (7) we can proceed as in the previous section. Equation (7) can be written in the equivalent form obtained by combining two equations of type (13) (one in the x -direction and one in the y -direction) and we can expand each of these in the form (28) (with appropriate changes in symbols). Thus combining these last two we obtain

$$\frac{1}{R}(\zeta_1 e^{-\frac{1}{2} R g_0 h} + \zeta_3 e^{\frac{1}{2} R g_0 h}) + \zeta_2 e^{\frac{1}{2} f_0 h} + \zeta_4 e^{-\frac{1}{2} f_0 h} - \left\{ 2 + \frac{2}{R} + \frac{h^2}{4} (f_0^2 + Rg_0^2) \right\} \zeta_0 = 0 \quad (37)$$

as a close approximation to (7), i.e. with neglect of terms in h^4 . To find expressions for pivotal values of u and v we proceed as in the last section by expanding $g(x, y_0)$ along the x -line 301 and likewise expanding $f(x_0, y)$ along the y -line 402. Substituting the corresponding first approximations found in this way into (34) this equation becomes

$$\frac{1}{R}(\zeta_1 e^{-\frac{1}{2} R g_0 h} + \zeta_3 e^{\frac{1}{2} R g_0 h}) + \zeta_2 e^{\frac{1}{2} f_0 h} + \zeta_4 e^{-\frac{1}{2} f_0 h} - \left(2 + \frac{2}{R} - h^2 \Phi_0 \right) \zeta_0 = 0, \quad (38)$$

which we can now compare with (37). We find that the equations agree exactly by virtue of the fact that $f(x, y) = \partial \psi / \partial x$, $g(x, y) = \partial \psi / \partial y$, so that the first two terms on the right-hand side of (36) cancel each other identically. From the point of view of the numerical analysis this is rather fortuitous; in the general case f and g may make an important contribution to (36) which our analysis suggests should not be ignored. The other general point which emerges is that even in Allen and Southwell's case a substantial improvement in accuracy should result from using the trapezoidal (or an even more accurate) formula to estimate pivotal values of u and v ; this would involve an almost negligible increase in labour.

5. This point leads us to consider whether Allen and Southwell's process could itself be modified to allow of this better approximation. As it stands the method replaces (8) by the equation (9); but if we *first* apply the transformation (17) to (8) and *then* approximate to the resulting equation (21) by the equation

$$Y'' + \phi_0 Y = r_0 \quad (39)$$

we would obtain

$$Y_1 + Y_3 - 2Y_0 \cos \phi_0^{\frac{1}{2}} h - \frac{2r_0}{\phi_0} (1 - \cos \phi_0^{\frac{1}{2}} h) = 0 \quad (40)$$

as an approximation to (21) at each nodal point and hence

$$\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - 2\zeta_0 \cos \phi_0^{\frac{1}{2}} h - \frac{2r_0}{\phi_0} (1 - \cos \phi_0^{\frac{1}{2}} h) = 0 \quad (41)$$

as a nodal approximation to (8), u having the significance of (20). This formula is quite different from either (12) or (23) although it agrees with the latter as far as terms in h^4 . It has two clear advantages over (12) in that we can extend u_1 and u_3 beyond the first approximations $u_1 = \frac{1}{2} p_0 h$, $u_3 = -\frac{1}{2} p_0 h$ (to which (12) is of necessity restricted) and, moreover, it takes proper account of the term involving p' (which occurs in ϕ). Applying this process to each of the equations (33) in turn and combining the results in the manner previously used, we obtain as a nodal approximation to (1) the equation

$$\alpha_0 (\zeta_1 e^{u_1} + \zeta_3 e^{u_3} - 2\zeta_0 \cosh \alpha_0^{\frac{1}{2}} h) / R(1 - \cosh \alpha_0^{\frac{1}{2}} h) + \beta_0 (\zeta_2 e^{v_2} + \zeta_4 e^{v_4} - 2\zeta_0 \cosh \beta_0^{\frac{1}{2}} h) / (1 - \cosh \beta_0^{\frac{1}{2}} h) = 0, \quad (42)$$

where

$$\alpha_0 = -\frac{1}{2} R \left(\frac{\partial g}{\partial x} \right)_0 + \frac{1}{4} R^2 g_0^2 \quad \left. \vphantom{\alpha_0} \right\} \quad (43)$$

and

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Here u and v have the significance of (35).

It would appear that (42) is a more correct application to (1) of Allen and Southwell's type of process than (7) and the amount of initial computation in using it would be substantially the same. But in deciding its merit relative to (34) an essentially new point is involved. It is whether we may in general expect the approximation (40) to represent (21) more accurately than customary finite differences. There is some evidence that this may be so for if ϕ and re^u were invariant with respect to x , (40) would represent (21) exactly. The new problem is quite a distinct one since the first-derivative term is now absent. In the second part of this paper we investigate this question, with interesting results, in relation to an important class of second-order equations.

6. Some applications to a class of eigenvalue problems

We shall consider the one-dimensional problem governed by the equation

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with extensions to its two-dimensional counterpart

$$\nabla^2 \zeta + \{\lambda\rho(x, y) - \sigma(x, y)\}\zeta = 0 \quad \left(\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad (45)$$

where in each case $\rho > 0$ throughout the domain of the problem and λ is a parameter possessing a discrete set of eigenvalues, with a corresponding set of modes for ζ , when ζ is required to satisfy specified boundary conditions (e.g. ζ vanishes on a simple closed boundary in the case of (45) or at the end points in the case of (44)). Both equations have physical importance in many fields.

For (44) the usual difference approximation at a nodal point 0 is

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and to find an approximation to a given mode we have to solve (46) for a set of nodal ζ -values together with the corresponding eigenvalue. If an initial approximation to the mode of (46) is known, relaxation methods can usually be employed satisfactorily for this purpose, the eigenvalue λ being estimated at any stage by the Rayleigh quotient which corresponds to (46), i.e.

$$\lambda h^2 = - \frac{\sum \{(\zeta_1 + \zeta_3 - 2\zeta_0)\zeta_0 - h^2 \sigma_0 \zeta_0^2\}}{\sum \rho_0 \zeta_0^2}, \quad (47)$$

where the summations extend over all ζ -values, including the end-points. This method is particularly satisfactory when $\zeta = 0$ at the end-points and the mode considered has no internal zeros (corresponding to the smallest λ). The solutions of (46) depend upon h and converge to the true solutions of (44) as h is decreased indefinitely. In a recent paper, Bolton and Scoins (4) have investigated the convergence in the special case $\rho(x) = 1$; (44) then becomes the one-dimensional Schrödinger equation with σ as the potential function. They show that if $\lambda = \Lambda(h)$ denotes the approximation obtained from (46) to a given true λ of (44), then

$$\Lambda(h) = \lambda + \nu h^2 + O(h^4) \quad (48)$$

and that if $\zeta = 0$ at the end-points and σ is bounded in $(0, 1)$ and non-singular at the end-points, then $\nu < 0$, i.e. for small enough h , $\Lambda(h) < \lambda$. We shall show that this result is valid for the more general case of (44) but first let us consider the modifications introduced if we apply Allen and Southwell's process to (44). The nodal approximation is

$$\zeta_1 + \zeta_3 - 2\zeta_0 \cos q_0^{\frac{1}{2}} h = 0 \quad (49)$$

which we can write approximately as

$$\zeta_1 + \zeta_3 - (2 - h^2 q_0 + \frac{1}{12} h^4 q_0^2) \zeta_0 = 0, \quad (50)$$

for small enough h , so that the order of difference between (46) and (49) is at once apparent. We can obtain a Rayleigh quotient corresponding to either (49) or (50); one can be obtained from the latter by assuming the term $h^4 q_0^2 \zeta_0 / 12$ to be temporarily independent of changes in λ . This gives

$$\lambda h^2 = - \frac{\sum [\zeta_1 + \zeta_3 - (2 + \frac{1}{12} h^4 q_0^2) \zeta_0] \zeta_0 - h^2 \sigma_0 \zeta_0^2}{\sum \rho_0 \zeta_0^2}. \quad (51)$$

Equations (50) and (51) can be used jointly in a relaxation process, the only essentially new point being that when we estimate λ from the quotient (51) the terms of order h^4 in the numerator would have to be computed from the λ -value determined by the previous cycle. If h is small enough these terms will be small; they constitute the difference between the eigenvalues of (46) and (50) and we can easily show that this is second order with respect to h . We shall consider only the case $\zeta = 0$ at the end-points. Let $\Lambda^{(1)}$ be the approximation to a given λ of (44) found from (46) and let $\Lambda^{(2)}$ be that found from (50), nodal ζ and q -values being similarly distinguished. If we multiply (46) by $\zeta_0^{(2)}$ and (50) by $\zeta_0^{(1)}$, subtract and sum over all nodal points we obtain

$$\sum [q_0^{(1)} - q_0^{(2)} + \frac{1}{12} h^2 \{q_0^{(2)}\}^2] \zeta_0^{(1)} \zeta_0^{(2)} = 0$$

and hence

$$\Lambda^{(2)} - \Lambda^{(1)} = \frac{\frac{1}{12} h^2 \sum \{q_0^{(2)}\}^2 \zeta_0^{(1)} \zeta_0^{(2)}}{\sum \rho_0 \zeta_0^{(1)} \zeta_0^{(2)}}. \quad (52)$$

Further, for small enough h , $\zeta_0^{(1)} \sim \zeta_0^{(2)}$ for every nodal point and $\rho_0 > 0$ by hypothesis so that ultimately $\Lambda^{(2)} > \Lambda^{(1)}$ for every mode and the result is true independently of h for the first (or one-signed) mode.

This result is interesting in view of Bolton and Scoins' result (48). If this latter were true under more general conditions it would mean that $\Lambda^{(2)}$ always corrects $\Lambda^{(1)}$ in the right direction, since $\Lambda^{(1)}$ is an underestimate of the true λ . We can show that this is so and, moreover, that $\Lambda^{(2)}$ is always an improvement.

7. Suppose that the range of integration, from $x = 0$ to $x = 1$ say, is divided into N equal intervals each equal to $h = 1/N$. We shall consider only the case $\zeta(0) = \zeta(1) = 0$ and suppose $q(x)$ to be bounded in $(0, 1)$ and non-singular at the end-points. It is convenient to write (15) in terms of successive derivatives of ζ at the point 0 rather than in terms of differences. If we then substitute in (44) we obtain the finite-difference relation

$$\lambda \rho_0 \zeta_0 = -N^2(\zeta_1 + \zeta_3 - 2\zeta_0) + \sigma_0 \zeta_0 + \frac{1}{12} h^2 \zeta_0^{iv} + \frac{1}{360} h^4 \zeta_0^{vi} + \dots \quad (53)$$

holding at all internal points, from which we obtain the Rayleigh quotient by multiplying by ζ_0 and summing for all internal points, i.e.

$$\lambda = \frac{\sum \{-N^2(\zeta_1 + \zeta_3 - 2\zeta_0) + \sigma_0 \zeta_0 + \frac{1}{2}h^2 \zeta_0'' + \frac{1}{360}h^4 \zeta_0^{(4)} + \dots\} \zeta_0}{\sum \rho_0 \zeta_0^2}. \quad (54)$$

If we suppose the higher derivatives to be expressed in terms of pivotal ζ -values and assume h to be small enough for (53) to converge, (54) defines λ as a function of the $N-1$ internal ζ -values. Its stationary values for all variations of the pivotal values, obtained by putting $\partial\lambda/\partial\zeta_0 = 0$ for each ζ_0 , give rise to (53) and are the true eigenvalues of (44). Neglecting terms in h^2 and higher powers of h in the numerator gives the quotient (47) and leads to the approximation $\lambda = \Lambda^{(1)}$. In its full form the quotient (54) is equivalent to the extended form of Bolton and Scoins' relation (48)† and we must now examine the sign of the terms in h^2 . In the interval (say) $x = x_0$ to $x = x_1$ we have, using trapezoidal summation,

$$\int_{x_0}^{x_1} \zeta(x) \zeta^{(iv)}(x) dx = \frac{1}{2}h(\zeta_0 \zeta_0^{(iv)} + \zeta_1 \zeta_1^{(iv)}) + O(h^3). \quad (55)$$

Summing (55) over each interval and remembering that $\zeta = 0$ at the end-points we obtain, approximately,

$$h \sum \zeta_0 \zeta_0^{(iv)} = \int_0^1 \zeta(x) \zeta^{(iv)}(x) dx = [\zeta(x) \zeta'''(x) - \zeta'(x) \zeta''(x)]_0^1 + \int_0^1 \{\zeta''(x)\}^2 dx. \quad (56)$$

The first term on the right hand of (56) vanishes by virtue of the boundary conditions since, by hypothesis, neither $\rho(x)$ nor $\sigma(x)$ is singular at an end-point. We can therefore replace the h^2 -sum in the numerator of (54) by the term

$$\frac{1}{12}h \int_0^1 \{\zeta''(x)\}^2 dx = \frac{1}{12}h \int_0^1 q^2 \zeta^2 dx, \quad (57)$$

using the differential equation (44). Since $\rho(x) > 0$, by hypothesis, it follows at once that (48) is true for a variable ρ and that $\nu < 0$. Moreover, replacing the integral (57) by its trapezoidal sum we can now write the quotient (54) as

$$\lambda = \frac{\sum \{-N^2(\zeta_1 + \zeta_3 - 2\zeta_0) + \sigma_0 \zeta_0 + \frac{1}{12}h^2 q_0^2 \zeta_0 + O(h^4)\} \zeta_0}{\sum \rho_0 \zeta_0^2} \quad (58)$$

and applying the variational condition to (58) leads to

$$\zeta_1 + \zeta_3 - (2 - h^2 q_0 + \frac{1}{12}h^4 q_0^2) \zeta_0 = O(h^6),$$

i.e. (56) is correct to a higher order of accuracy than (46). This result seems

† Summing over all pivotal values effectively reduces each power of h in the numerator of (54) by one, but the sum in the denominator is of order $1/h$, so the equivalence follows.

particularly useful since if we first compute an eigenvalue and mode of (46) we can at once get an improved eigenvalue by substituting our computed mode in (51), i.e. without any further differencing of the solution. The mode can then be improved using (50) but subsequent changes are unlikely to alter the eigenvalue by very much. Incidentally we may notice in passing that although we have approximated to (49) by (50), this is not necessary. We can use (49) in conjunction with its proper Rayleigh quotient, which only involves replacing the factor $2+h^4q_0^2/12$ in (51) by $2\cos q_0^1 h + h^2 q_0$.

Example 2

To test these results we have applied them to an example previously solved by Fox (op. cit. 184) who uses it to illustrate the 'difference-correction' method of standard finite differences. The problem is that of finding the mode of the Mathieu equation

$$\zeta'' + (\lambda - 2\cos 2x)\zeta = 0 \quad (59)$$

which vanishes at $x = 0$ and $x = \pi$ and has no internal zero in the interval $(0, \pi)$; this mode is symmetrical about $x = \frac{1}{2}\pi$. Fox uses an interval $h = \frac{1}{8}\pi$ and obtains a first approximation according to (46), i.e. in this case

$$\zeta_1 + \zeta_3 - (2 - \lambda h^2 + 2h^2 \cos 2x_0)\zeta_0 = 0, \quad (60)$$

estimating λ from the corresponding Rayleigh quotient. His final result for the eigenvalue is $\lambda h^2 = -0.0210$ which, to this number of figures, is the correct eigenvalue of (60). This is then improved by differencing the corresponding approximation for ζ and adding difference corrections according

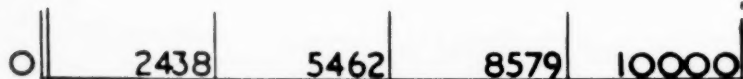


FIG. 2

to (15), leading eventually to $\lambda h^2 = -0.017008$, i.e. $\lambda = -0.11029$, which is known to be the correct eigenvalue of (59) to four decimals. The underestimate given by the eigenvalue of (60) is clearly substantial in this case (almost 24 per cent. of the true value) and we have found that this result can be improved considerably if we use (50) instead. The eigenvalue obtained from this set of equations is $\lambda h^2 = -0.01712$, i.e. $\lambda = -0.1110$ and the corresponding eigenvector is given in Fig. 2. This lies between the eigenvector obtained from (60) and Fox's final corrected eigenvector and

although it is not nearly as accurate as the latter it leads, as we expect from the properties of the Rayleigh quotient, to a much improved eigenvalue (now not much more than $\frac{1}{2}$ per cent. in error).

8. We can apply Allen and Southwell's treatment of (1) to (45) except that there is not now a unique way of separating (45) into its two component equations. The simplest method is to write this equation in the form

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{2} q \zeta = A(x, y), \quad \frac{\partial^2 \zeta}{\partial y^2} + \frac{1}{2} q \zeta = -A(x, y), \quad (61)$$

where again $q = \lambda p - \sigma$. We can now approximate to the first equation of (61) along the x -line 301 by the equation

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{1}{2} q_0 \zeta = A_0,$$

which leads to the approximation

$$\zeta_1 + \zeta_3 - 2\zeta_0 \cos[(\frac{1}{2}q_0)h] - \frac{4A_0}{q_0} \{1 - \cos[(\frac{1}{2}q_0)h]\} = 0. \quad (62)$$

Dealing similarly with the second of (61) along the y -line 402 we obtain

$$\zeta_2 + \zeta_4 - 2\zeta_0 \cos[(\frac{1}{2}q_0)h] + \frac{4A_0}{q_0} \{1 - \cos[(\frac{1}{2}q_0)h]\} = 0, \quad (63)$$

so that if we add (62) to (63) we get the approximation

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 4\zeta_0 \cos[(\frac{1}{2}q_0)h] = 0. \quad (64)$$

This result can be closely represented, for small enough h , by the approximation

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - (4 - h^2 q_0 + \frac{1}{24} h^4 q_0^2) \zeta_0 = 0 \quad (65)$$

and, as for (44), it differs by a term in h^4 from the standard finite-difference formula

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - (4 - h^2 q_0) \zeta_0 = 0. \quad (66)$$

Two other obvious ways of separating (45) into component equations are the arrangements

$$\frac{\partial^2 \zeta}{\partial x^2} = B(x, y), \quad \frac{\partial^2 \zeta}{\partial y^2} + q \zeta = -B(x, y) \quad (67)$$

and

$$\frac{\partial^2 \zeta}{\partial x^2} + q \zeta = C(x, y), \quad \frac{\partial^2 \zeta}{\partial y^2} = -C(x, y). \quad (68)$$

Each of these gives a nodal approximation which is different from (64) and neither of these approximations is symmetrical with respect to nodal ζ -values, although we ought really to expect this symmetry on account of the symmetry of (45) with respect to the second derivatives. However, we

can combine the approximations obtained from (67) and (68) into the single approximation

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 4\zeta_0\{2 + (h^2q_0 - 2)\cos q_0^{\frac{1}{2}}h\}/\{2 + h^2q_0 - 2\cos q_0^{\frac{1}{2}}h\} = 0 \quad (69)$$

and although (69) is not the same (or as simple) as (64) it leads to (65) if we expand in powers of h and retain only terms in h^4 and below. This lends more confidence to the use of (65) as a satisfactory approximation.

The Rayleigh quotient corresponding to (65) is

$$\lambda h^2 = - \frac{\sum [\{\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - (4 + \frac{1}{2}h^4q_0^2)\zeta_0 - h^2\sigma_0\zeta_0^2\}]}{\sum \rho_0\zeta_0^2}, \quad (70)$$

which differs only by the term in h^4 from the quotient corresponding to (66). If $\zeta = 0$ on the boundaries of any rectangular region of integration we can show, using a method similar to that used for (44), that the corresponding form of (52) is now

$$\Lambda^{(2)} - \Lambda^{(1)} = \frac{\frac{1}{24}h^2 \sum \{q_0^{(2)}\}^2 \zeta_0^{(1)} \zeta_0^{(2)}}{\sum \rho_0 \zeta_0^{(1)} \zeta_0^{(2)}}, \quad (71)$$

where $\Lambda^{(1)}$ now refers to an eigenvalue of (66) and $\Lambda^{(2)}$ to the corresponding eigenvalue of (65). Moreover, for the case of rectangular boundaries, Bolton and Scoins (5) have shown that (48) (Λ here being $\Lambda^{(1)}$ in our notation) is equally applicable to the two-dimensional Schrödinger equation and that $\nu < 0$ if $\zeta = 0$ on the boundaries. This is again interesting since (71) shows that, for small enough h , $\Lambda^{(2)} > \Lambda^{(1)}$ by a term of order h^2 . An investigation along the lines of the previous section now proves much more difficult, however, and we shall not attempt it. Instead we shall suggest a somewhat tentative justification of the superiority of (65) by considering the results of two numerical examples. Both relate to the transverse vibrations of membranes, in which $\sigma = 0$ in (45) and ρ represents the surface density of the membrane, ζ being the transverse deflexion.

Example 3

The first relates to a rectangular membrane with sides in the ratio 2:3. The surface density is constant and the membrane is fixed ($\zeta = 0$) along its edges. This case has previously been considered by Allen (6) as an example to illustrate the use of standard relaxation methods. We shall suppose the units to be non-dimensional, the length of the shorter side being unity. Taking the origin at one corner of the rectangle with the y -axis along the shorter side, the appropriate governing equation is

$$\nabla^2 \zeta + \lambda \zeta = 0 \quad (72)$$

and the exact solutions for the modes are

$$\zeta = \sin\left(\frac{2m\pi x}{3}\right)\sin(n\pi y) \quad (73)$$

with eigenvalues

$$\lambda = \pi^2(\frac{1}{9}m^2 + n^2), \quad (74)$$

where m and n are any integers. The first mode ($m = n = 1$) possesses symmetry about both $x = \frac{2}{3}$ and $y = \frac{1}{2}$ and is the one which Allen has computed approximately. Using an interval $h = \frac{1}{3}$ and the standard formula (66) (with $q_0 = \lambda$) he obtains the final value $\lambda = 14.14$, which underestimates the correct answer by rather less than 1 per cent. We

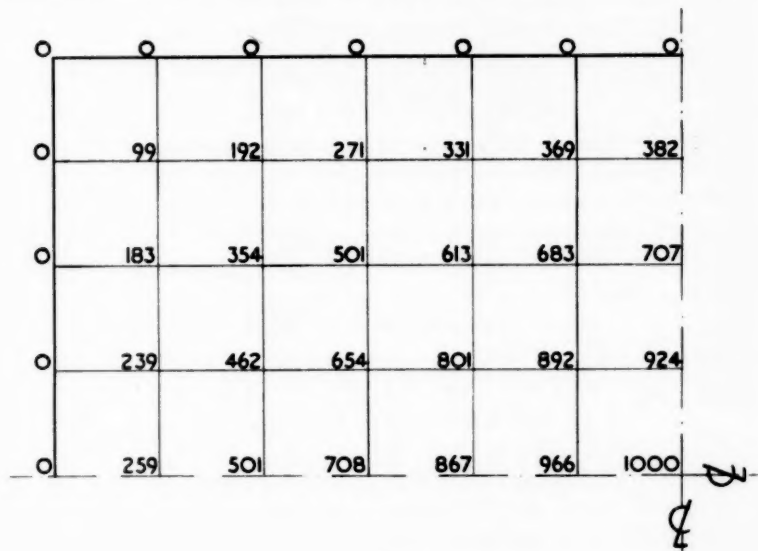


FIG. 3

solved the same problem using (65) and our final answer on the same interval is $\lambda = 14.24$; this is again an underestimate but of an amount little more than 0.1 per cent. The corresponding mode is given in Fig. 3 (the maximum deflexion being fixed at 1,000) and differs nowhere by more than a unit from the appropriate mode of (73) (when scaled to the same maximum deflexion).

Example 4

Our second example relates to a square membrane with a variable surface density, previously treated by Allen, Fox, Motz, and Southwell (7) by standard methods. With coordinate axes as before the appropriate non-dimensional governing equation is

$$\nabla^2 \zeta + 2\lambda(1 - x - y + 2xy)\zeta = 0, \quad (75)$$

ζ vanishing on the unit square bounded by the axes and the lines $x = 1$,

$y = 1$. Using an interval $h = \frac{1}{16}$, Southwell *et al.* have computed the mode with no internal nodal lines using (66); their final accepted eigenvalue is $\lambda = 19.5661^\dagger$ and they state that they would expect (from consideration of similar computations when the density is constant and the exact eigenvalue known) that the true λ should lie between 19.54 and 19.60. As our first test we substituted their final computed mode into (70) and found that it resulted in an estimate $\lambda = 19.62881$. Residuals were then calculated from (65) and it was found necessary to modify the initial assumption for the mode by not more than one unit at a few nodal points to arrive at an acceptable solution. Substitution of this in (70) gave $\lambda = 19.62878$, which is our final value. To see whether our value is an improvement in this case we have obtained an independent solution to the problem.

9. The process we use is essentially that of the method of variation of parameters. The modes of (75) satisfy $\zeta = 0$ on the unit square and can therefore be expressed as an absolutely and uniformly convergent series

$$\zeta = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \sin i\pi x \sin j\pi y, \quad (76)$$

where i and j are positive integers. If we substitute in the expression

$$U = \int_0^1 \int_0^1 \zeta \nabla^2 \zeta \, dx dy + 2\lambda \int_0^1 \int_0^1 (1-x-y+2xy)\zeta^2 \, dx dy \quad (77)$$

and put $\partial U / \partial a_{mn} = 0$ for all integer values of m and n , i.e. we use the property that (75) is satisfied at a stationary value of (77), we obtain the doubly-infinite set of algebraic equations

$$\{\pi^2(m^2+n^2)-\lambda\}a_{mn} = \frac{64\lambda mn}{\pi^4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\{1-(-1)^{i+m}\}\{1-(-1)^{j+n}\}ij}{(i^2-m^2)^2(j^2-n^2)^2} a_{ij} \quad (i \neq m, j \neq n), \quad (78)$$

any equation of the set being obtained by assigning a pair of integer values to m and n . The characteristic solutions of (78) are the true modes of (75) expressed in the form (76); we cannot solve (78) exactly but we can approximate to a given mode to any desired accuracy by standard relaxation methods. For example, we are looking for the mode with no internal nodal lines and a good approximation ignores all but the first term of (76), giving from (78)

$$(2\pi^2-\lambda)a_{11} = 0,$$

i.e. $\lambda = 2\pi^2$, with the corresponding mode

$$\zeta = \sin \pi x \sin \pi y.$$

[†] Subsequently amended to 19.564 by Miss G. Vaisey (see Southwell (8)); we worked in relation to the original paper cited.

We can improve this by taking account of the next important term in (76) (judged to be the one whose coefficient is associated with the next smallest factor on the left-hand side of (78)). In fact this is a_{22} since it is obvious that (78) consists of two *independent* sets of equations involving firstly only those a_{ij} for which $i+j$ is even and secondly only those for which $i+j$ is odd. Working to five decimals we then obtain the equations

$$(2\pi^2 - \lambda)a_{11} - 0.12978\lambda a_{22} = 0,$$

$$0.12978\lambda a_{11} - (8\pi^2 - \lambda)a_{22} = 0,$$

from which we obtain a new estimate $\lambda = 19.62981$ with the mode

$$\xi = \sin \pi x \sin \pi y + 0.0429 \sin 2\pi x \sin 2\pi y.$$

By extending this process to sufficient equations we have computed the mode and eigenvalue to high accuracy, i.e. we have included all $a_{ij} > 0.00005$ and taken into account enough equations to ensure that no a_{ij} we have omitted has any effect on computed values. The equations were solved by relaxation methods and the eigenvalue estimated from the corresponding Rayleigh quotient. It is easily shown that $a_{ij} \equiv a_{ji}$ and our values are given, correct to four decimals, in Table 2.

TABLE 2. *Values of a_{ij}*

$i \backslash j$	1	2	3	4	5	6
1	1.0000		0.0016		0.0001	
2		0.0431		0.0012		0.0002
3	0.0016		0.0009		0.0001	
4		0.0012		0.0001		
5	0.0001		0.0001			
6		0.0002				

The eigenvalue is $\lambda = 19.62884$ and is the true eigenvalue of (75) to all the figures quoted. Returning to the relaxation solutions it follows that Southwell *et al.* are incorrect in their prediction of the bounds for λ and that our value underestimates the true value by about three parts in a million, a result of quite surprising accuracy. We could add the general comment that, as far as we can say from these examples, there is a definite advantage in using Allen and Southwell's type of approximation to solve equations of type (45). For the final part of our paper we shall return to equations with the first derivative present and make an extension of our previous results which is of more especial interest to ordinary differential equations, for reasons that we now explain.

10. An extension of Numerov's method

The question of the desirability of using improved finite-difference formulae (i.e. with error terms of a *higher order* with respect to h) in the

solution of partial differential equations such as (1) or (45) is debatable. Such formulae require the introduction of more nodal points than are present in Fig. 1 and the relaxation process is more complicated. But for second-order ordinary equations in the normal form

$$\zeta'' + q(x)\zeta = r(x) \quad (79)$$

a more accurate *three-point* formula is obtainable. It follows from (15) that

$$h^2(1 + \frac{1}{12}\delta^2)\zeta_0'' = (\delta^2 + \frac{1}{240}\delta^6 - \frac{13}{15120}\delta^8 + \dots)\zeta_0 \quad (80)$$

and we can use (79) to express $\delta^2\zeta_0''$ in terms of pivotal values of $r - q\zeta$ so that a finite-difference equivalent of (79) is

$$(1 + \frac{1}{12}h^2q_1)\zeta_1 + (1 + \frac{1}{12}h^2q_3)\zeta_3 - (2 - \frac{5}{8}h^2q_0)\zeta_0 - \frac{1}{12}h^2(r_1 + 10r_0 + r_3) = 0 \quad (81)$$

with an error of only $\frac{1}{240}\delta^6\zeta_0$. Equation (81) is generally ascribed to Numerov (9) but it cannot be applied in this way when the first-derivative term is present (as in (8)) because $\delta^2\zeta_0''$ would now involve $\delta^2\zeta_0'$ and more pivotal points must be introduced. Our previous results suggest how to overcome this. We first transform (8) to (21), apply Numerov's method to this latter equation and then return to our original variable as before. This gives the pivotal approximation

$$(1 + \frac{1}{12}h^2\phi_1)\zeta_1 e^{u_1} + (1 + \frac{1}{12}h^2\phi_3)\zeta_3 e^{u_3} - (2 - \frac{5}{8}h^2\phi_0)\zeta_0 - \frac{1}{12}h^2(r_1 e^{u_1} + 10r_0 + r_3 e^{u_3}) = 0 \quad (82)$$

as an accurate three-point representation of (8). By the term 'three-point' here, of course, we mean involving only three pivotal values of ζ . If the functions ϕ and u are expressed analytically the formula is three-point in every sense; but if these functions must be expressed approximately in terms of pivotal values we shall need extra pivotal points (certainly for finding ϕ_1 and ϕ_3). This can always be done *ab initio* and does not alter the relaxation process at all.

Example 5

To test this result we have again used a problem solved by Fox (3, 73), i.e. to find an approximation to the solution of

$$\zeta'' + \zeta' + 4x\zeta + 1 = 0 \quad (83)$$

with end conditions $\zeta(0) = \zeta(1) = 0$. Using an interval $h = \frac{1}{4}$, Fox first obtains an approximation according to the standard formula

$$(1 + \frac{1}{2}h)\zeta_1 + (1 - \frac{1}{2}h)\zeta_3 - (2 - 4h^2r_0)\zeta_0 + h^2 = 0 \quad (84)$$

which he then improves by calculating difference corrections and adding them in according to the full formulae (14) and (15). We give both his solutions in Table 3, the corrected final solution being not more than a unit out in the last decimal from the true solution. We have obtained this

accuracy directly using (82). We also give, for interest, solutions according to (12) and (23), although we do not expect either of these to give results comparable with (82). Allen and Southwell's formula gives a better result than (23) but probably only, because the term involving p' in ϕ (which the former fails to take account of) is identically zero in this case. In any case, both give better results than (84).

TABLE 3

x	ζ (L. Fox)		ζ	ζ	ζ
	Eqn. (84)	Corrected	Eqn. (82)	Eqn. (12)	Eqn. (23)
0.0	0.0	0.0	0.0	0.0	0.0
0.25	0.1236	0.1218	0.1218	0.1213	0.1223
0.50	0.1573	0.1541	0.1541	0.1540	0.1557
0.75	0.1104	0.1075	0.1076	0.1080	0.1094
1.00	0	0	0	0	0

11. Conclusions

The investigation of this paper was undertaken to discover why Allen and Southwell's method of approximating to (1) should be superior to standard approximations. There can be little doubt that it is, for it deals more effectively with the terms involving first derivatives. But we have shown that (1) is rather a special case since the coefficients of the first-derivative terms are related so that a term which may otherwise be important cancels identically. That this term can be important is clearly indicated in example 1; the investigation here relates to the ordinary differential equation (8) but the essential features are the same. In fact, cancellation of the term in question (arising from p') is not possible for ordinary differential equations unless as is exceptional, $p(x)$ is constant, as in example 5. The investigation has led us to a number of alternative approximations which, as far as we can see, possess all the features of the removal of the first-derivative term from the equation, i.e. they behave with respect to convergence as if this term were absent (a case in which it has always been known that finite-difference methods give more satisfactory results). In the case of ordinary differential equations we can extend the basic method of approximation to take account of higher differences without introducing more pivotal points (an extension previously limited to equations with the first-derivative term absent).

The central section of the paper, which deals with the eigenvalue problems governed by (44) and (45), involves questions of a rather different nature since now no first-derivative terms are present in the equations. The essential point here is whether Allen and Southwell's process still gives improved results. Subject to certain limitations (some of which could,

perhaps, be removed), we have shown that the approximation to (44) is certainly improved. The fact that the error term in (50) is of order h^6 is of some interest in view of the fact that the derivation of the approximation (12), from which it is ultimately obtained, does not require ζ to possess derivatives beyond the second order. This is distinct from the improved order of accuracy which would result from applying Numerov's approximation (81) since this requires higher derivatives of ζ to exist, although the distinction is a theoretical rather than a practical one. In the case of the partial differential equation (45) our investigation is more tentative. Even if we restrict ourselves to the case $\zeta = 0$ on a rectangular boundary, it is more difficult to prove that (65) is a more accurate approximation than (66). The evidence of the numerical example suggests, however, that it is. Consider, for example, the problem relating to the transverse deflexion of a uniform rectangular membrane ($\sigma = 0$, $\rho = 1$ in (45)). In this case (71) reduces at once to

$$\Lambda^{(2)} = \Lambda^{(1)} + \frac{1}{24}h^2\{\Lambda^{(1)}\}^2,$$

which we can write in the form

$$\Lambda^{(2)} = \Lambda^{(1)} + \frac{1}{24}h^2\{\Lambda^{(1)}\}^2 + O(h^4). \quad (85)$$

In the particular example† (cf. Southwell (8), 383) of a uniform membrane bounded by a unit square it is known that, for the fundamental mode, $\Lambda^{(1)} = 19.72347$ when $h = \frac{1}{32}$. Neglecting terms in h^4 in (85) we obtain $\Lambda^{(2)} = 19.73930$ as against the exact value $\lambda = 2\pi^2 = 19.73921$, to five decimals. Again, if we consider the membrane with variable density $\rho(x, y)$ and suppose that in (71) we can approximate to the pivotal ρ -values by some mean value $\bar{\rho}$ (since, in effect, the right-hand side of (71) is an average), then we can approximate to $\Lambda^{(2)}$ by the relation

$$\Lambda^{(2)} \doteq \Lambda^{(1)} + \frac{1}{24}h^2\bar{\rho}\{\Lambda^{(1)}\}^2. \quad (86)$$

An obvious choice for $\bar{\rho}$ would be the integrated mean value, i.e.

$$\bar{\rho} = \frac{\int_D \rho(x, y) \, dx dy}{\int_D dx dy},$$

where D is the domain bounded by the membrane. In example 4, for instance, this gives $\bar{\rho} = 1$ and, using Miss Vaisey's value $\Lambda^{(1)} = 19.564$ when $h = \frac{1}{16}$, we obtain $\Lambda^{(2)} = 19.626$ from (86), a very great improvement obtained with no substantial additional labour.

In general the use of Allen and Southwell's type of approximation raises very wide issues, of course, e.g. when we may best expect a function to be

† I am indebted to a referee for pointing out this example.

represented more adequately by expansions other than polynomials. Such general questions cannot readily be answered and this paper provides only a limited amount of evidence that may be worth consideration in specific cases.

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SOLVING AN ALGEBRAIC EQUATION BY DETERMINING HIGH POWERS OF AN ASSOCIATED MATRIX USING THE CAYLEY-HAMILTON THEOREM

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SUMMARY

This paper describes a method of solving an algebraic equation by determining high powers of an associated matrix using the Cayley-Hamilton theorem. The method described will be found to have practical applications to linear equation analogue computers for finding the zeros of polynomials.

1. Introduction

ALTHOUGH various numerical methods are available in literature (1, 2, 3) for finding the zeros of polynomials, not much attention has been paid so far to the application of matrix methods of solution (viz. converting the problem to an eigenvalue problem) which has been suggested by Aitken (4). Also the methods of finding zeros of polynomials in all the analogue computers constructed so far have been direct (5), using non-linear and trigonometric function potentiometers. It is the object of this paper to point out that Aitken's method (4) can be readily mechanized with great advantage either in linear equation computers in which matrix multiplications can be accomplished (Barker (6)), or in direct secular-equation solvers (5). This, together with the acceleration process herein suggested, results in the significant improvement of the method of determination of real or complex roots of equations, with either real or complex coefficients.

2. The matrix method

It is well known (7) that any polynomial of degree n

$$a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + x^n = 0 \quad (1)$$

has the associated companion matrix ($n \times n$)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 & -a_0 \\ 1 & 0 & \dots & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad (2)$$

whose eigenvalues are the roots of (1). Hence, the problem reduces to the determination of eigenvalues of A for which a variety of methods are already available (1, 2, 3, 8). Among these, the matrix power method is the one that is the easiest and quickest, for the determination of the eigenvalue of A with largest modulus.

The matrix power method essentially consists in repeatedly multiplying an arbitrary column vector $X^{(0)}$ by the matrix A so that the recurrence relation

$$AX^{(m-1)} = X^{(m)} \quad (3)$$

is satisfied.

For purposes of computation it is desirable to reduce the largest of the components of $X^{(m)}$ to unity and each time this normalized vector is fed back for getting the new iterated vector $X^{(m+1)}$. Therefore when this process converges (to a desired accuracy) at the end of m cycles, we have

$$AX^{(m)} = \lambda_1 X^{(m+1)}, \quad (4)$$

where $X^{(m)}$ is almost identical with $X^{(m+1)}$ and λ_1 is the root with largest modulus. This method, as is easily seen, thus involves the successive multiplications by the matrix A , say m times. Even this labour could be considerably reduced by a technique suggested in the next section.

Once the largest eigenvalue is known, the other eigenvalues are obtained either by using, (1) the deflation method, (2) Richardson's purification process, (3) the Gram-Schmidt process described in the references cited above, or by reducing the polynomial itself to one of a lower degree by synthetic division and again solving the 'reduced equation' by applying the matrix power method. However, the rounding-off errors in these processes have to be carefully considered when high accuracy is aimed at.

Once all the roots are obtained to a sufficient approximation, they can be further refined either by the Bairstow-Lin process (3) or Jahn's (9) method:

3. The method of acceleration

Since we know by the Cayley-Hamilton theorem that any matrix A should satisfy its own characteristic equation, we have by (1)

$$-(a_0 I + a_1 A + \dots + a_{n-1} A^{n-1}) = A^n. \quad (5)$$

Therefore any higher power of A , say A^m ($m > n$) is always expressible in terms of a matrix polynomial in A , involving only the first $(n-1)$ powers of A . So by choosing m to be large, quick convergence can be obtained to the eigenvector X_1 corresponding to the largest of the eigenvalues λ_1 , just by a single multiplication of an arbitrary vector by A^m , which is known, provided an initial algebraic calculation is made to

express \mathbf{A}^m in terms of \mathbf{A}^r [where $r = 1, 2, \dots, (n-1)$]. When the eigenvector \mathbf{X}_1 is thus known, a single multiplication by \mathbf{A} could yield λ_1 also.

The computation of \mathbf{A}^r does not present serious difficulties because of the inherent nature of the companion matrix \mathbf{A} (even though it may be difficult for any general matrix). The peculiarity with \mathbf{A} is that all the first $(n-1)$ columns of \mathbf{A}^2 are the same as the second to the n th columns of \mathbf{A} , all the first $(n-1)$ columns of \mathbf{A}^3 are the same as the second to the n th columns of \mathbf{A}^2 , and so on. Consequently it is necessary to determine only $(n-2)$ columns for getting \mathbf{A}^r ($r = 2, 3, \dots, n-1$). Thus, even if no analogue computer is available, these multiplications could be carried out with great ease, using desk calculators.

4. Example

These ideas will be clearer if a simple example is considered, namely, the determination of the largest root of the familiar equation

$$x^3 - x - 1 = 0. \quad (6)$$

Here

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

and $\mathbf{A}^3 = \mathbf{A} + \mathbf{I}$ by (5). Therefore,

$$\begin{aligned} \mathbf{A}^{18} &= [(\mathbf{A}^3)^3]^2 = 37\mathbf{A}^2 + 49\mathbf{A} + 28\mathbf{I} \\ &= \begin{bmatrix} 28 & 37 & 49 \\ 49 & 65 & 86 \\ 37 & 49 & 65 \end{bmatrix}. \end{aligned}$$

Taking a general arbitrary vector $\mathbf{X}^{(0)} = (1, 1, 1)$,

$$\mathbf{A}^{18}\mathbf{X}^{(0)} = 200 (0.570, 1.000, 0.755) = 200 (\mathbf{X}^{(18)}).$$

Therefore

$$\mathbf{A}\mathbf{X}^{(18)} = 1.325 (0.570, 1.000, 0.755).$$

The value of λ_1 is 1.325 (the exact value accurate to 4 figures . . . 1.3247). If still higher powers of \mathbf{A} are used much higher accuracy can be obtained.

5. Conclusion

The method described above has several limitations. If the eigenvalues of (2) are all distinct and real, there is no difficulty in applying the method. However, it is possible that the dominant eigenvalues are either (a) nearly equal, or (b) complex pairs (even if \mathbf{A} is real), or (c) roots of equal modulus (real or complex), or (d) repeated (real or complex) roots.

For the case (a), the convergence can be obtained without any difficulty taking a high enough power of \mathbf{A} .

But in the case of (b), the matrix multiplication of an arbitrary real vector $\mathbf{X}^{(0)}$ would not lead to a convergent sequence (even if each pair is well separated from the next root (real or complex) and the elements of iterates would oscillate instead of tending to a limit. From a knowledge of three successive iterates (in the case of well separated roots) or equally spaced iterates (if several roots are quite close), e.g. $\mathbf{A}^m \mathbf{X}^{(0)}$, $\mathbf{A}^{m+\Delta} \mathbf{X}^{(0)}$, and $\mathbf{A}^{m+2\Delta} \mathbf{X}^{(0)}$ where Δ is any properly chosen integer, it is possible to determine the dominant complex pair of roots (10). Instead of having an arbitrary real vector, if a complex arbitrary vector is chosen, the pair of roots can be obtained directly. In such a case, however, the largest component of iterated vector must each time be normalized to $(1+i.0)$ and iterations performed in the usual way.

For the case (c) when there are real roots of equal modulus the situation is practically analogous to case (b), but the successive iterates $\mathbf{A}^m \mathbf{X}^{(0)}$, $\mathbf{A}^{m+1} \mathbf{X}^{(0)}$, etc., will be alternately parallel to one of the two eigenvectors and hence the multiplicity will be revealed. In this case it is always possible to add a constant real matrix $\mu \mathbf{I}$ to the matrix \mathbf{A} , thus altering the moduli of the real roots from $\pm\lambda$ to $(\pm\lambda+\mu)$, and hence separating them. The same method can also be applied to separate real and complex roots having the same moduli.

For the case (d), there are certain difficulties of convergence which are intimately connected with the properties of matrix (2). If the matrix (2) is diagonalizable, the repeated roots will have an independent set of eigenvectors associated with each one of them, and taking any general arbitrary vector the convergence will not be on one of the eigenvectors, but will only be a vector which is a linear combination of both these vectors. However, there will not be any difficulty of convergence in such a case.

But in the case of repeated eigenvalues (real or complex), it can happen that the number of linearly independent vectors associated with this eigenvalue is less than the multiplicity. In this case the matrix cannot be diagonalized and the convergence to a limit will be extremely slow. Unfortunately there are no simple and direct methods of testing the diagonalizability of a matrix. In such a case, therefore, one has either to go to very high powers of the matrix \mathbf{A} and use Aitken's δ^2 -process (3) or to use a method developed by the author which is akin to Graeffe's technique. The latter method will be published elsewhere.

The one great advantage of the matrix method is that it breaks up the polynomial into a set of linear equations, so that a linear equation analogue computer would serve the purpose better than a polynomial computer, which involves a number of non-linear computing elements (5).

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CONTENTS

G. PEOOTS and L. SOWERBY: Axially Symmetric Stagnation Point Flow with Heat Transfer in Magnetohydrodynamics	385
A. N. ERGUN: Axisymmetric Solutions of the Incompressible Magnetohydrodynamic Equations	408
J. R. JONES: Flow of a Non-Newtonian Liquid in a Curved Pipe	428
K. WALTERS: The Motion of an Elastico-Viscous Liquid contained between Coaxial Cylinders (II)	444
ARNOLD D. KERR: Uniformly Stretched Plates subjected to Concentrated Transverse Forces	462
M. R. OSBORNE: On the Propagation of Sound in a Layered Fluid Medium	472
S. C. R. DENNIS: Finite Differences associated with Second-order Differential Equations	487
E. V. KRISHNAMURTHY: Solving an Algebraic Equation by determining High Powers of an Associated Matrix using the Cayley-Hamilton Theorem	508

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